Siegel-Weil Formula and Average of Certain Conformal Field Theories

Yongchang Zhu

HKUST

July 24, 2024

I will discuss the application of the **Siegel-Weil formula** to establish a case of the **holographic duality**, which connects a 2-dimensional conformal field theory to a gravitational theory in 3 dimension.

I will discuss the application of the **Siegel-Weil formula** to establish a case of the **holographic duality**, which connects a 2-dimensional conformal field theory to a gravitational theory in 3 dimension.

I will mainly follow the recent works by two groups of physicists:

A. Maloney and E. Witten,

N. Afkhami-Jeddi, H. Cohn, T. Hartman, and A. Tajdini

Their works (2021) provide evidence that the **Narain conformal field theory** is dual to an unconventional theory of gravity in three dimensions with $U(1)^D \times U(1)^D$ symmetries.

Their works (2021) provide evidence that the **Narain conformal field theory** is dual to an unconventional theory of gravity in three dimensions with $U(1)^D \times U(1)^D$ symmetries.

We will present our computations regarding the correlation functions of Narain Conformal Field Theory.

Siegel-Weil Formula

Fix a positive integer $D > 2$. Let

$$\mathbb{R}^{D,D} = \mathbb{R}^D \times \mathbb{R}^D$$

be the vector space with the symmetric bilinear form

$$((u_1, u_2), (v_1, v_2)) = u_1 \cdot v_1 - u_2 \cdot v_2,$$

Siegel-Weil Formula

Fix a positive integer $D > 2$. Let

$$\mathbb{R}^{D,D} = \mathbb{R}^D \times \mathbb{R}^D$$

be the vector space with the symmetric bilinear form

$$((u_1, u_2), (v_1, v_2)) = u_1 \cdot v_1 - u_2 \cdot v_2,$$

where $u_1 \cdot v_1$ and $u_2 \cdot v_2$ are the standard inner product in $\mathbb{R}^D$.

Let $O(D, D, \mathbb{R})$ be the isometry group.

An **even unimodular lattice** $L \subset \mathbb{R}^{D,D}$ is a free $\mathbb{Z}$-module spanned by a basis such that

$$(u, v) \in \mathbb{Z}, \quad (u, u) \in 2\mathbb{Z}, \quad \Lambda^* = \Lambda.$$

for $u, v \in \Lambda$.

We fix an even unimodular lattice L_0.

An **even unimodular lattice** $L \subset \mathbb{R}^{D,D}$ is a free $\mathbb{Z}$-module spanned by a basis such that

$$(u, v) \in \mathbb{Z}, \quad (u, u) \in 2\mathbb{Z}, \quad \Lambda^* = \Lambda.$$

for $u, v \in \Lambda$.

We fix an even unimodular lattice L_0.

The group

$$O(D, D, \mathbb{Z}) = \{g \in O(D, D, \mathbb{R}) \,|\, gL_0 = L_0\}$$

is a discrete subgroup. The space $O(D, D, \mathbb{R})/O(D, D, \mathbb{Z})$ has a finite invariant measure, which we normalize to be 1.

The groups $O(D, D, \mathbb{R})$ and $Sp_{2m}(\mathbb{R})$ form a dual pair in Sp_{4Dm}.

We have the Weil representation $L^2(\mathbb{R}^{D,D} \otimes \mathbb{R}^m)$ of MP_{4Dm}.

The groups $O(D, D, \mathbb{R})$ and $Sp_{2m}(\mathbb{R})$ form a dual pair in Sp_{4Dm}.

We have the Weil representation $L^2(\mathbb{R}^{D,D} \otimes \mathbb{R}^m)$ of MP_{4Dm}.

The theta functional

$$\theta : \in \mathcal{S}(\mathbb{R}^{D,D} \otimes \mathbb{R}^m) \to \mathbb{C}, \quad \theta(f) = \sum_{r \in L_0 \otimes \mathbb{Z}^m} f(r)$$

is $O(D, D, \mathbb{Z}) \times SP_{2m}(\mathbb{Z})$-invariant.

For $f \in \mathcal{S}(\mathbb{R}^{D,D} \otimes \mathbb{R}^m)$, the Siegel-Weil formula is the identity

$$\int_{g \in O(D,D,\mathbb{R})/O(D,D,\mathbb{Z})} \theta(\pi(g)f)dg = \sum_{h \in P(\mathbb{Z}) \backslash SP_{2m}(\mathbb{Z})} (\pi(h)f)(0)$$

Example. Take $m = 1$, $f \in \mathcal{S}(\mathbb{R}^{D,D})$ is given by

$$f(x, y) = e^{\pi i \tau x \cdot x} e^{\pi i (-\bar{\tau}) y \cdot y}.$$

where τ is in the upper half plane.

For $g \in O(D, D, \mathbb{R})$,

$$\begin{aligned} \theta(\pi(g)f) &= \sum_{r=(\alpha,\beta)\in gL_0} e^{\pi i \tau \alpha\cdot\alpha} e^{\pi i (-\bar{\tau}) \beta\cdot\beta} \\ &= \sum_{r=(\alpha,\beta)\in gL_0} q^{\frac{1}{2}\alpha\cdot\alpha} \bar{q}^{\frac{1}{2}\beta\cdot\beta} \end{aligned}$$

where

$$q = e^{2\pi i \tau}, \qquad \bar{q} = e^{2\pi i (-\bar{\tau})}$$

For $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$(\pi(h)f)(0) = |c\tau + d|^{-D}$$

Siegel-Weil formula for this f is

$$\int_{O(D,D,\mathbb{R})/O(D,D,\mathbb{Z})} \sum_{r=(\alpha,\beta)\in gL_0} q^{\frac{1}{2}\alpha\cdot\alpha}\bar{q}^{\frac{1}{2}\beta\cdot\beta} dg$$
$$= \sum_{(c,d)=1} |c\tau + d|^{-D}$$

Rational Conformal Field Theory and Vertex Operator Algebras

A rational conformal field theory corresponds to a rational vertex operator algebra.

Rational Conformal Field Theory and Vertex Operator Algebras

A rational conformal field theory corresponds to a rational vertex operator algebra.

A vertex operator algebra is a graded vector space

$$V = \oplus_{n=0}^{\infty} V_n$$

where each element $a \in V$ has a corresponding vertex operator $a(z)$ (acting on V), also known as a field associated with a. These operators satisfy certain axioms.

A vertex operator algebra is rational if it satisfies certain finiteness properties.

There are two special elements 1 (vacuum) and ω (Virasora element) in V such that

$$1(z) = 1$$

$$\omega(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}$$

L_n satisfies the Virasoro algebra relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c$$

L_0 action gives the gradation. i.e, $L_0|_{V_n} = n$.

Given a rational VOA V and a Riemann surface Σ, one can construct the partition function and correlation functions on Σ.

$$((a_1, z_1), \ldots, (a_n, z_n))_\Sigma$$

If Σ has genus one, the correlation functions can be computed by taking traces.

We will always set $q = e^{2\pi i\tau}$, where τ is the upper half plane, so $|q| < 1$.

$$\Sigma = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau = \mathbb{C}^*/q^{\mathbb{Z}}$$

The correlation functions on Σ is

$$q^{-\frac{c}{24}} \mathrm{Tr}\, a_1(z_1) \cdots a_n(z_n) q^{L_0},$$

which is a Jacobi modular form with weights that depend on $a_1, \ldots, a_n$.

An historically important example of a vertex operator algebra is the Moonshine module V, constructed by Frenkel-Lepowsky-Meurman, which is a rational vertex operator algebra with central charge $c = 24$.

An historically important example of a vertex operator algebra is the Moonshine module V, constructed by Frenkel-Lepowsky-Meurman, which is a rational vertex operator algebra with central charge $c = 24$.

$$q^{-1} Trq^{L_0} = J(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + \dots$$

If $a \in V$ satisfies

$$L_n a = 0 \quad \text{for} \quad n \geq 1, \quad L_0 a = ma.$$

The 1 point correlation function

$$q^{-1} Tr\, a(z) q^{L_0}$$

is a modular form of weight $2m$ for $SL_2(\mathbb{Z})$.

There are countably many rational vertex operator algebras.
However, the classification is currently out of reach.

There are countably many rational vertex operator algebras.
However, the classification is currently out of reach.

The task of classifying rational vertex operator algebras includes the classification of even positive definite lattices and simple Lie algebras.

There are countably many rational vertex operator algebras.
However, the classification is currently out of reach.

The task of classifying rational vertex operator algebras includes the classification of even positive definite lattices and simple Lie algebras.

In fact, one can construct a rational VOA using an even positive definite lattice.

One can construct a rational VOA using a simple Lie algebra and a given positive integer.

This talk focuses on **non-rational** 2D conformal field theory. There doesn't seem to be an official definition of 2D conformal theory, but it can be identified when it arises.

VOAs play a limited role in non-rational 2D conformal field theory.

It does NOT dictate the theory as in the rational case.

The partition function and correlation functions are defined on each Riemann surface Σ.

Narain Conformal Field Theory and the Moduli Space

Given an even unimodular lattice L in $\mathbb{R}^{D,D}$, Narain constructed a non-rational conformal field theory (1986)

The space of states is

$$
\begin{aligned}
F_L = \quad & \mathbb{C}[x_l^i(-k) : i = 1, \dots, D, k = 1, 2, \dots] \\
& \otimes \mathbb{C}[x_r^i(-k) : i = 1, \dots, D, k = 1, 2, \dots] \\
& \otimes \mathbb{C}[L]
\end{aligned}
$$

On F_L, there are multiplication operators $x_l^i(-k), x_r^i(-k)$, and differentiation operators

$$\frac{\partial}{\partial x_l^i(-k)}, \quad \frac{\partial}{\partial x_r^i(-k)}$$

They satisfy the Heisenberg algebra relations.

The Heisenberg algebra generates a VOA (which is independent of L).

Also there are two copies of Virasoro algebras (by Sugawara construction) acting on V_L:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}D$$
$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}D.$$

$$L_0 x_l^i(-n) = n\, x_l^i(-n), \quad L_0 x_r^i(-n) = 0$$

$$\bar{L}_0 x_r^i(-n) = n\, x_r^i(-n), \quad \bar{L}_0 x_l^i(-n) = 0$$

$$L_0 x_l^i(-n) = n\, x_l^i(-n), \quad L_0 x_r^i(-n) = 0$$

$$\bar{L}_0 x_r^i(-n) = n\, x_r^i(-n), \quad \bar{L}_0 x_l^i(-n) = 0$$

For $\alpha = (a, b) \in L$,

$$L_0 e^\alpha = \frac{1}{2} a \cdot a\, e^\alpha, \quad \bar{L}_0 e^\alpha = \frac{1}{2} b \cdot b\, e^\alpha$$

The conformal field theory V_L and $V_{L'}$ are equivalent iff $L' = gL$ for some $g \in O(D) \times O(D) \subset O(D, D)$.

The moduli of all Narain conformal field theories is

$$O(D) \times O(D) \backslash O(D, D, \mathbb{R}) / O(D, D, \mathbb{Z})$$

As a conformal field theory, it has a partition function for each Riemann surface.

For the genus 0 Riemann surface, the partition function is 1.

For genus one Riemann surface $\Sigma = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, the partition function is

$$Z(L,\tau) = q^{-\frac{D}{24}} \bar{q}^{-\frac{D}{24}} \mathrm{Tr} q^{L_0} \bar{q}^{\bar{L}_0}$$

$$Z(L,\tau) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{(a,b)\in\Lambda} q^{\frac{1}{2}(a,a)} \bar{q}^{\frac{1}{2}(b,b)} = \frac{1}{|\eta(\tau)|^{2D}} \Theta_L(\tau) \qquad (1)$$

where $\eta(\tau)$ is the Dedekind eta function given by

$$\eta(\tau) = q^{-\frac{1}{24}} \Pi_{n=1}^{\infty}(1-q^n).$$

For

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{2}$$

The Riemann surfaces

$$\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$$

and

$$\mathbb{C}/\mathbb{Z} + \mathbb{Z}\frac{a\tau + b}{c\tau + d}$$

are isomorphic, so we should have

$$Z(L, \frac{a\tau + b}{c\tau + d}) = Z(L, \tau). \tag{3}$$

This can be checked directly by the properties

$$\Theta_L\left(\frac{a\tau+b}{c\tau+d}\right) = |c\tau+d|^D\Theta_L(\tau)$$

$$|\eta\left(\frac{a\tau+b}{c\tau+d}\right)|^{2D} = |c\tau+d|^D|\eta(\tau)|^{2D}$$

For a Riemann surface Σ of genus $g \geq 1$, the partition function $Z(L, \Sigma)$ is given by

$$Z(L, \Sigma) = \frac{1}{\Phi(\Sigma)} \sum_{(\alpha,\beta)\in L^g} e^{\pi i \alpha \Omega \alpha^T - \pi i \beta \Omega \beta^T},$$

where $\Phi(\Sigma)$ is independent of the lattice L, Ω is the period matrix of Σ, which lives in the Siegel's upper half space $\mathcal{H}_g$.

Holographic Duality

The holographic duality relates a 2-dimensional conformal field theory to a gravitational theory in 3 dimensions.

For a 2-d CFT, the corresponding 3-d is usually not known.

A. Maloney and E. Witten, N. Afkhami-Jeddi, H. Cohn, T. Hartman, and A. Tajdini proposed the 3-d theory corresponding to the Narain conformal field theories.

The average of all Narain CFT corresponds to the

$U(1)_D \times U(1)_D$ Chern-Simons theory on a three dimensional manifold with boundary.

3D Theory with $U(1)^D \times U(1)^D$ Symmetry

The action of the theory is

$$S(A, \bar{A}) = \frac{i}{8\pi}\int_M A_I \wedge dA_I - \bar{A}_I \wedge d\bar{A}_I + \frac{1}{16\pi}\int_{\partial M} d^2x\sqrt{g}g^{ab}A_a^I A_b^I.$$

Here, g_{ab} is the metric on ∂M.

The basic fields are 1-forms $A^1, \dots, A^D, \bar{A}^1, \dots, \bar{A}^D$,

The partition function is formally defined as

$$Z_3(M, g) = \int_{A^I, \bar{A}^I} e^{iS}\mathcal{D}A\mathcal{D}\bar{A}.$$

We consider the following (M, g):

We first take

$$\partial M = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$$

with the flat metric. We write ∂M as

$$\mathbb{R} + \mathbb{R}\tau/\mathbb{Z} + \mathbb{Z}\tau = \mathbb{R}/\mathbb{Z} \times \mathbb{R}\tau/\mathbb{Z}\tau$$

Replace $\mathbb{R}/\mathbb{Z} = S^1$ be the 2d disc.

M is a handlebody (a donut).

For the above (M, g), we have

$$Z_3(M, g) = \frac{1}{|\eta(\tau)|^{2D}}$$

Note that M and g on ∂M depends only on τ, we denote

$$Z_3(M, g)$$

by

$$Z_3(\tau).$$

Theorem (Maloney-Witten, Afkhami-Jeddi-Cohn-Hartman-Tajdini, 2021)

The average of $Z(L, \tau)$ over L is

$$\sum_{h \in P(\mathbb{Z}) \backslash SL_2(\mathbb{Z})} Z_3(h\tau)$$

that is,

$$\int_{O(D) \times O(D) \backslash O(D,D,\mathbb{R}) / O(D,D,\mathbb{Z})} Z(gL_0, \tau) dg = \sum_{h \in P(\mathbb{Z}) \backslash SL_2(\mathbb{Z})} Z_3(h\tau)$$

This formula is a direct consequence of Siegel-Weil formula.

$$\begin{aligned}
&\int_{O(D)\times O(D)\backslash O(D,D,\mathbb{R})/O(D,D,\mathbb{Z})} Z(gL_0,\tau)dg \\
&= \int \frac{1}{|\eta(\tau)|^{2D}} \sum_{(a,b)\in gL_0} q^{\frac{1}{2}(a,a)}\bar{q}^{\frac{1}{2}(b,b)}dg \\
&= \frac{1}{|\eta(\tau)|^{2D}} \int \sum_{(a,b)\in gL_0} q^{\frac{1}{2}(a,a)}\bar{q}^{\frac{1}{2}(b,b)}dg \\
&= \frac{1}{|\eta(\tau)|^{2D}} \sum_{(c,d)=1} |c\tau+d|^{-D}
\end{aligned}$$

The right hand side is

$$\begin{aligned}
&\sum_{h\in P(\mathbb{Z})\backslash SL_2(\mathbb{Z})} Z_3(h\tau) \\
&= \sum_{h\in P(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \frac{1}{|\eta(h\tau)|^{2D}} \\
&= \frac{1}{|\eta(\tau)|^{2D}} \sum_{(c,d)=1} |c\tau + d|^{-D}
\end{aligned}$$

The summation

$$\sum_{g\in P(\mathbb{Z})\backslash SL_2(\mathbb{Z})} Z_3(g\tau)$$

is interpreted as the sum over all the handlebodies with boundary $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$.

We write the torus as $S^1 \times S^1$, filling in the first factor by a two-dimensional disc D_2, we get the handlebody $D_2 \times S^1$.

We can construct a more general handlebody by taking other decompositions of the torus as $S^1 \times S^1$, and fills in the first factor by a disc.

These additional handlebodies are labeled by elements of the modular group $SL_2(\mathbb{Z})$.

Each handlebody is uniquely identified by an element of the coset $P(\mathbb{Z})\backslash SL_2(\mathbb{Z})$, since elements in $P(\mathbb{Z})$ does not create a new handlebody.

Therefore

$$\sum_{h \in P(\mathbb{Z}) \backslash SL_2(\mathbb{Z})} Z_3(h\tau)$$

is interpreted as the sum of the partition functions over all handlebodies with boundary as $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$.

For fixed genus one Riemann surface $\Sigma = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, the average Narian CFT on Σ is equal to the sum of the partition functions over handbodies with Σ as a boundary.

Average of Flavored Partition Functions

S. Datta, S. Duary, P. Kraus, P. Maity and A. Maloney considered the the flavored partition functions (2022):

$$Z_L(\tau, \bar{\tau}, z_L, z_R)$$

where $z_L = (z_1, \ldots, z_D), z_R = (\bar{z}_1, \ldots, \bar{z}_D)$.

$$Z_L(\tau, \bar{\tau}, z, \bar{z}) = q^{-\frac{D}{24}} \bar{q}^{-\frac{D}{24}} Trq^{L_0} \bar{q}^{\bar{L}_0} e^{2\pi i \sum_{k=1}^{D} z_k h_k(0) - 2\pi i \sum_{k=1}^{D} \bar{z}_k \bar{h}_k(0)} \tag{4}$$

$$Z_L(\tau,\bar\tau,z,\bar z) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{(a,b)\in\Lambda} q^{\frac{1}{2}(a,a)} \bar q^{\frac{1}{2}(b,b)} e^{2\pi i(a,z)-2\pi i(b,\bar z)} \tag{5}$$

We have for $g \in SL_2(\mathbb{Z})$,

$$\begin{aligned} & Z_L\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar\tau+b}{c\bar\tau+d}, \frac{z_L}{c\tau+d}, \frac{z_R}{c\bar\tau+d}\right) \\ &= exp\left(\frac{\pi i c z_L^2}{c\tau+d} - \frac{\pi i c z_R^2}{c\bar\tau+d}\right) Z_L(\tau,\bar\tau,z,\bar z) \end{aligned}$$

They computed the average

$$\int_{L\in\mathcal{M}_D} Z_L(\tau,\bar{\tau},z,\bar{z})d\mu(L)$$

which is equal to

$$\int_{\mathcal{M}_D} Z_L(\tau,z)d\mu(L) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{(c,d)=1} \frac{1}{|c\tau+d|^D} e^{-\pi i\left(\frac{cz_l^2}{c\tau+d}-\frac{cz_R^2}{c\bar{\tau}+d}\right)}$$

The formula can be proved by applying the Siegel-Weil formula to

$$f(\tau, z^L, z^R, x, y) = f_{\tau, z^L, z^R}(x, y) = e^{\pi i \tau (x,x)} e^{\pi i (-\bar{\tau})(y,y)} e^{2\pi i (z^L, x)} e^{-2\pi i (z^R, y)}$$

The result confirms the proposed duality.

Correlation Functions of Vertex Operators

On V_L, we have actions of the two copies of Heisenberg algebra.

$$a(n), \quad a \in \mathbb{R}_L^D, n \in \mathbb{Z}$$

$$[a_1(m), a_2(n)] = a_1 \cdot a_2 m \delta_{m+n,0}$$

$$\bar{a}(n), \quad \bar{a} \in \mathbb{R}_R^D, n \in \mathbb{Z}$$

$$[\bar{a}_1(m), \bar{a}_2(n)] = -\bar{a}_1 \cdot \bar{a}_2 m \delta_{m+n,0}$$

The vertex operators

$$a(z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}$$

$$\bar{a}(z) = \sum_{n\in\mathbb{Z}} \bar{a}(n) z^{-n-1}$$

The correlation function of $a_1(z_1), \dots, a_n(z_n)$ on $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ is

$$(q\bar{q})^{-\frac{N}{24}} \mathrm{Tr}\, a_1(z_1) \cdots a_n(z_n) q^{L_0} q^{\bar{L}_0}$$

It can be computed, the result can be written as a sum of functions parametrized by partitions of type

$$\{a_1, \dots, a_m\} = J \cup \{a_{i_1}, a_{j_1}\} \cup \dots \{a_{i_m}, a_{j_m}\}$$

J is any subset with $m - |J|$ even. J is distinguished.

For each partition as above, the corresponding function is

$$(a_{i_1}, a_{j_1}) \cdots (a_{i_m}, a_{j_m}) \wp_2(z_{i_1} - z_{j_1}) \cdots \wp_2(z_{i_m} - z_{j_m}) D(J) Z_L(\tau, z_L, z_R)|_{z_L = Z_R = 0}$$

The average over L can also be computed using Siegel-Weil formula.

It suggests how to define the correlation functions for the three dimension theory.

Thank you!