The Relative Langlands Duality

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July 21, 2024

(Joint with Bryan P.J. Wang)

Local Langlands Program

- F local field,
- $\mathbf{G}$ reductive group over F; set $G = \mathbf{G}(F)$;
- $\mathrm{Irr}(G) = \{$isom. classes of irred. smooth reps of $G\}$.

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where

- An L-parameter is a $G^\vee$-conjugacy class of maps

  $$\phi : WD_F = W_F \times \mathrm{SL}_2 \longrightarrow G^\vee$$

  where $G^\vee$ is the Langlands dual group of G.
- $\eta \in \mathrm{Irr}(S_\phi)$, where S_ϕ is an appropriate component group.

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For a given ϕ, have associated L-packet

$$\Pi_\phi = \{\pi_\eta : \eta \in \mathrm{Irr}(S_\phi)\}.$$

A-parameters and A-packets

On the other hand, there is a notion of A-parameters:

$$\psi : WD_F \times \mathrm{SL}_2 \longrightarrow G^\vee$$

with associated A-packet

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where each π_η is a unitary representation of finite length.
One can associate an L-parameter to ψ via:

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right), \quad \text{for } w \in WD_F.$$

One has

$$\Pi_{\phi_\psi} \subset \Pi_\psi.$$

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Expect: $\mathrm{Irr}_{H,\chi}(G)$ corresponds to L-parameters which factor through some $J^\vee \to G^\vee$. So (H, χ)-dist. reps are functorial lifts from another group J.

Classical Examples

Periods	(G, H, χ)
Whittaker	$G \supset U$ (maximal unipotent) $\chi = \psi$ generic
Symplectic	$\mathrm{GL}_{2n} \supset \mathrm{Sp}_{2n}$
Shalika	$\mathrm{GL}_{2n} \supset P_{n,n} \supset \mathrm{GL}_n^{\Delta} U$ $\chi = 1_{\mathrm{GL}_n} \otimes \psi(\mathrm{Tr}(-))$
Basic Gross-Prasad	$\mathrm{SO}_{2n} \times \mathrm{SO}_{2n+1} \supset \mathrm{SO}_{2n}^{\Delta}$
General GP $n < m$ opp. parity	$\mathrm{SO}_n \times \mathrm{SO}_m \supset \mathrm{SO}_n^{\Delta} U$ $1 \otimes \chi$ generic

Questions and Issues

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Upshot The subject of RLP is based on a number of examples but there is no systematic framework,

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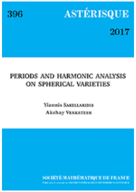
Upshot The subject of RLP is based on a number of examples but there is no systematic framework, that is, until the publication of the Asterisque volume [SV]:



Relative Langlands according to [SV]

Basic Objects	spherical G-variety $X = H\backslash G$
Problem	spectral decomposition of $L^2(X)$ or $C^\infty(X)$
Dual Data	(i) $\iota_X : X^\vee \times \mathrm{SL}_2 \to G^\vee$ (ii) (graded symplectic) representation V_X of $X^\vee$
Conjectural Answer	H-dist. π have A-parameters factoring via ι_X: $WD_F \times \mathrm{SL}_2 \to X^\vee \times \mathrm{SL}_2 \to G^\vee$

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Key Innovations of [SV]:

- Providing a uniform framework for the RLP;
- Definition of the dual data $(X^\vee, \iota, V_X)$ in terms of which the conjectural answer can be formulated.

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- there are certain natural G-modules which are multiplicity-free and whose spectral decomposition can be described in the style of [SV], but which nonetheless do not fall into the framework of [SV].

An example of this last point is the theta correspondence, i.e. the spectral decomposition of the Weil representation under the action of a dual pair.

BZSV

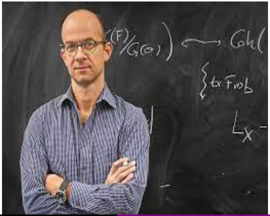
The above issues are better addressed in the 400-page preprint [BZSV] of Ben-Zvi, Sakellaridis and Venkatesh:

RELATIVE LANGLANDS DUALITY

DAVID BEN-ZVI, YIANNIS SAKELLARIDIS AND AKSHAY VENKATESH

ABSTRACT. We propose a duality in the relative Langlands program. This duality pairs a Hamiltonian space for a group G with a Hamiltonian space under its dual group $\check{G}$, and recovers at a numerical level the relationship between a period on G and an L-function attached to $\check{G}$; it is an arithmetic analog of the electric-magnetic duality of boundary conditions in four-dimensional supersymmetric Yang–Mills theory.

This is a draft. We anticipate making another round of changes before submitting it for publication. All comments are very welcome! In particular, if we have failed to attribute or properly reference a work it is most likely due to either ignorance or forgetfulness - please tell us.



Relative Langlands according to [BZSV]

Objects	Hyperspherical Hamiltonian G-variety M
Problem	spectral decomp. of quantization Π_M of M
Dual Data	Hyperspherical Hamiltonian $G^\vee$-variety $M^\vee$
Conjecture	Galois action has fixed point on $M^\vee_{slice}$

Comparing [BZSV] with [SV]

	[SV]	[BZSV]
Objects	spherical X	hyperspherical M
Problem	$L^2(X)$	Quantization of M
Dual Data	$(X^\vee, \iota_X, V_X)$	hyperspherical $M^\vee$
Conj.	Factor though ι_X	Galois-fixed points on $M^\vee_{slice}$

Hyperspherical Hamiltonian G-varieties M

(a) Hamiltonian G-variety: M is a symplectic G-variety with a G-equivariant moment map

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$$T_x\mathcal{O} \subset T_xM \text{ satsifies } T_x\mathcal{O}^\perp \subset T_x\mathcal{O}.$$

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Example:

- a symplectic vector space W with $G = \mathrm{Sp}(W)$-action;
- If X is a spherical G-variety (affine smooth), then $M = T^*X$ is hyperspherical.

Structure Theory of Hyperspherical Varieties

Suppose one is given the initial data:

- $\iota : H \times \mathrm{SL}_2 \longrightarrow G$, with $H \subset Z_G(\iota(\mathrm{SL}_2))$ a spherical subgroup;
- S a (graded) symplectic (finite dim) representation of H.

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Observe that the above data is of the same type as the dual data from [SV]:

$$\iota_X : X^\vee \times \mathrm{SL}_2 \longrightarrow G^\vee$$

and V_X a (graded symplectic) rep. of $X^\vee$. The process of Whittaker induction thus produces a hyperspherical $G^\vee$-variety $M^\vee$.

Symplectic Reduction and Induction

- Symplectic reduction of M with respect to G:

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$$M = S \times_{\mathfrak{h}^*}^H T^*(G) = (S \times_{\mathfrak{h}^*} \mathfrak{g}^*) \times^H G.$$

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Example:

$$T^*(H\backslash G) = \{0\} \times_{\mathfrak{h}^*}^H T^*(G)$$

is the symplectic induction of the H-space $\{0\}$ to G, and also the symplectic reduction of $T^*(G)$ with respect to H.

Whittaker Induction of a H-space S along $\iota : H \times \mathrm{SL}_2 \to G$

Let $\{h, e, f\}$ be $\mathfrak{sl}_2$-triple associated to $\iota|_{\mathrm{SL}_2}$. Then h induces a grading on $\mathfrak{g}$, with

$$\mathfrak{u} = \bigoplus_{i>0} \mathfrak{g}_i \supset \mathfrak{u}^+ = \bigoplus_{i>1} \mathfrak{g}_i \quad \text{and } \mathfrak{h} \subset \mathfrak{g}_0.$$

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The symplectic vector space $\mathfrak{u}/\mathfrak{u}^+$ is a Hamiltonian HU-space with (shifted) moment map

$$\mu : \mathfrak{u}/\mathfrak{u}^+ \xrightarrow{\kappa} (\mathfrak{u}/\mathfrak{u}^+)^* \longrightarrow \mathfrak{u}^*$$

given by

$$\mu(u) = \kappa(u) + f.$$

Whittaker Induction

Then

$$M = (S \times \mathfrak{u}/\mathfrak{u}^+) \times^{HU}_{\mathfrak{h}^* + \mathfrak{u}^*} T^*G.$$

is the symplectic induction of $S \times \mathfrak{u}/\mathfrak{u}^+$ from HU to G.

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A special case is when $S = 0$. Then

$$M = (f + \mathfrak{g}^e/\mathfrak{h}) \times^H G.$$

3 Basic Examples

M	Data for construction
$T^*(H\backslash G)$	$\iota : H \to G$, $S = 0$
$T^*_e(U\backslash G)$	$\iota : \mathrm{SL}_2 \to G$ (regular SL_2), $S = 0$
$(W, G \subset \mathrm{Sp}(W))$	$\iota : G \to G$, $S = W$

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Mixed Example: $S = 0$ and

$$\iota : \mathrm{GL}_n \times \mathrm{SL}_2 \longrightarrow \mathrm{GL}_{2n} \quad \text{(tensor product).}$$

Then

$$M = \{ \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix} : B \in M_n \} \times^{\mathrm{GL}_n^{\Delta}} \mathrm{GL}_{2n} = T_e(\mathrm{GL}_n^{\Delta} U \backslash \mathrm{GL}_{2n})$$

Call this the Shalika variety since it gives rise to the Shalika period.

Quantization

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- to a symplectic variety M, one can attach an associated Hilbert space Π_M.
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One does not have this, but many standard notions and constructions in symplectic geometry can be quantized, i.e. have natural representation theoretic counterparts, as realized by Kirillov, Guillemin-Sternberg, Kazhdan etc.

Classical vs. Quantization

Classical	Quantum
M	(ρ_M, V_M)
Coisotropic	Multiplicity-free
$S \times_{\mathfrak{h}^*}^{H} T^*G$ (Symplectic induction)	$\mathrm{Ind}_H^G \rho_S$ (induction)
$M \times_{\mathfrak{g}^*}^{G} \{0\}$ (Symplectic reduction)	$(V_M)_G$ (G-coinvariants)

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The last line is often expressed as:

Quantization commutes with Reduction.

Examples of Quantization

M	Π_M
$T^*(X)$, X affine smooth spherical	$L^2(X)$
$T_e^*(U\backslash G) = G \times^U (e + \mathfrak{u}^\perp)$ $= (e + \mathfrak{g}^f) \times G$ (e regular nilpotent)	$L^2(U, \psi\backslash G)$ (Whittaker/Gelfand-Graev module)
symplectic vector space $W = X + X^*$	Weil representation $L^2(X)$
$(V \otimes W, \mathrm{O}(V) \times \mathrm{Sp}(W))$	Theta correspondence for $\mathrm{O}(V) \times \mathrm{Sp}(W)$

Summary

	[SV]	[BZSV]
Objects	spherical X	hyperspherical M
Spectral Qn	$L^2(X)$	Quantization of M
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What is gained from [SV] to [BZSV]:

- Scope of RLP expanded (e.g. to include theta correspondence);
- there is now a clear symmetry between the basic object M and the basic dual data $M^\vee$.

The A-model associated to M

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- Variant: if ψ is an A-parameter of G, set

$$A_M(\psi) = \sum_{\eta \in \mathrm{Irr}(S_\psi)} A_M(\psi, \eta)$$

with

$$A_M(\psi, \eta) = \dim \mathrm{Hom}_G(\pi_\eta, \Pi_M).$$

Thus the A-model captures the spectral decomposition problem we are studying.

The B-model

Given an A-parameter

$$\psi : WD_F \times \mathrm{SL}_2 \longrightarrow G,$$

one obtains an $\mathfrak{sl}_2$-triple (e_ψ, h_ψ, f_ψ) with associated Slodowy slice

$$\Sigma_\psi := f_\psi + \mathfrak{g}^{e_\psi} \subset \mathfrak{g} \cong \mathfrak{g}^*.$$

Set

$$M_\psi = \mu_M^{-1}(\Sigma_\psi) \subset M.$$

Via $\phi_\psi \times |-|$, WD_F acts on M_ψ. We would like to define:

$$B_M(\psi) = \sum_{\eta \in \mathrm{Irr}(S_\psi)} B_M(\psi, \eta)$$

with

$$B_M(\psi, \eta) = \sum_{x \in M_\psi^{WD_F}} \ldots\ldots\ldots\ldots$$

Duality Conjecture

The above discussion led [BZSV] to the following

Conjecture

There is an involutive duality

$$\{\text{hyperspherical } G\text{-var.}\} \longleftrightarrow \{\text{hyperspherical } G^\vee\text{-var.}\}$$

such that if $M \longleftrightarrow M^\vee$, *then*

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In particular,

$$A_M(\psi) \neq 0 \Longrightarrow (M^\vee)_\psi^{WD_F} \neq \emptyset.$$

Examples of Duality [BZSV]

M	$M^\vee$
point	$T_e(U\backslash G)$
$T^*(X)$	$V_X \times^{X^\vee} G^\vee$
$T^*(\mathrm{Sp}_{2n}\backslash \mathrm{GL}_{2n})$ (symplectic period)	$T_e(\mathrm{GL}_n^\Delta U\backslash \mathrm{GL}_{2n})$ (Shalika period)
$T^*(SO_{2n}^\Delta\backslash(\mathrm{SO}_{2n}\times \mathrm{SO}_{2n+1}))$ (Basic Gross-Prasad)	$V_{2n}\otimes W_{2n}$ (Equal Rank Theta Corr.)
$(V\otimes W, \mathrm{O}(V)\times \mathrm{Sp}(W))$	General GP-varieties

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We consider special cases of the data:

$$(\iota_X : H \times \mathrm{SL}_2 \to G, S)$$

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The associated G-variety M just depends on a unipotent conjugacy class $e \in G$. The quantization of M_e is a generalized Whittaker/Gelfand-Graev G-module:

$$\Pi_e = \mathrm{Ind}_{H \cdot U}^{G} 1_H \otimes \psi.$$

Results I

Our first result addresses the question: for which e is M_e hyperspherical? Note that for classical groups, nilpotent orbits are classified by partitions or Young diagrams with parity constraints (plus extra data).

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Proposition

Assume $G = \mathrm{O}_{2n}$ for simplicity. Then M_e is hyperspherical if and only if e belongs to the following list:

- $e = [2n - r, 1^r]$, *r odd (hook type)*
- $e = (2^n)$ *(Shalika type)*
- $e = (3, 3)$, $(4, 4)$ *or* $(6, 6)$ *(sporadic type)*

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Theorem

For $e \in G = \mathrm{O}_{2n}$ of hook type.

$$M_e^\vee = M_{e^\vee},$$

where $e^\vee \in G^\vee = \mathrm{O}_{2n}$ is also a nilpotent element of hook type. More precisely, the relation $e \longleftrightarrow e^\vee$ is depicted by the following diagram.

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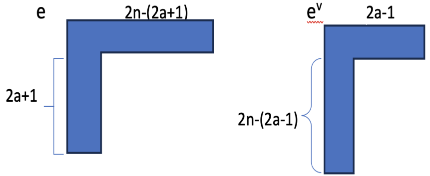

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- The proof of the theorem involves resolving two spectral decomposition problems (for M and $M^\vee$ resp.) , and showing that the answer can be described in terms of the dual variety ($M^\vee$ and M resp.) as dictated by the BZSV conjecture.

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- In particular, we use the results of Gomez and Zhu on the transfer of generalized Whittaker models under theta correspondence.
- Bryan has extended the results of Gomez-Zhu to the global setting and the L^2-setting, allowing us to resolve the L^2 and global version of the BZSV conjecture in our context.

Results of Gomez-Zhu

Theorem (Gomez-Zhu)

Consider the dual pair $\mathrm{O}_{2n} \times Sp_{2n-2a}$. *Given*

- $\pi \in \mathrm{Irr}(\mathrm{O}_{2n})$ *with big theta lift* $\Theta(\pi)$ *on* Sp_{2n-2a}*;*
- *a unipotent class* e *of* Sp_{2n-2a} *with associated generalized Whittaker datum* $(H_e U_e, \psi_e)$*,*
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where e' *is obtained from* e *as follows:*

- *remove the first column from the Young diagram of* e;
- *add boxes to the first column of what's left.*

Moreover, $(H_e, H_{e'})$ *forms a "dual pair", so that* $\Theta(\sigma)$ *is the big theta lift of* σ *on* $H_{e'}$.

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In other words,

Hyperspherical Duality "commutes" with Reduction.

Happy 60th Birthday, Chengbo!