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MVW-involutions for p-adic groups

1. MVW-involutions
2. GSpin(V) and GPin(V)
3. Application to multiplicity one.

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① MVW-Involution

MVW = Moeglin – Vigneras – Waldspurger.

F = p-adic field of char $F = 0$

G = reductive group over F

Let $\sigma : G \to G$ be an involution (i.e. $\sigma^2 = id$, $\sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2)$)

For each smooth rep. (π, V), define (π^σ, V) by $\pi^\sigma(g) = \pi(\sigma(g))$.

<u>Note</u>: $\pi \mapsto \pi^\sigma$ is a covariant exact functor.

<u>Def</u>: an involution $\sigma : G \to G$ is called an <u>MVW-involution</u> if for $\pi \in Irr(G)$, $\pi^\sigma = \pi^\vee$

<u>Note</u>: $\pi \mapsto \pi^\vee$ is contravariant exact functor

<u>Thm</u>: An MVW-involution exists for $G = GL_n, O(V), SO(V), Sp_{2n}, \widetilde{Sp}_{2n}, U(V)$

① <u>GL_n</u>: $\sigma(g) = {}^t g^{-1}$

② <u>$O(V)$</u>. $\sigma(g) = g$, i.e. $\sigma = id$. so $\pi = \pi^\vee$ for $O(V)$

③ <u>$SO(V)$</u>: If $\dim V = 2k-1$ (k = any) or $2k$ w/ k = even, $\sigma = id$

If $\dim V = 2k$ w/ k = odd, then $\sigma(g) = s g s^{-1}$, $s \in O(V) \setminus SO(V)$.

④ Sp_{2n}: Let $s = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} \in GSp_{2n}$. $\sigma(g) = s g s^{-1}$

⑤ <u>$\widetilde{Sp}_{2n}$</u>:

$$\begin{tikzcd}
\{\pm1\} \arrow[r, phantom, "="] \arrow[d] & \{\pm1\} \arrow[d] \\
\widetilde{Sp}_{2n} \arrow[r, "\tilde\sigma"] \arrow[d] & \widetilde{Sp}_{2n} \arrow[d] \\
Sp_{2n} \arrow[r, "\sigma"'] & Sp_{2n}
\end{tikzcd}$$

$\exists!$ lift $\tilde\sigma$ of σ.

Then $\tilde\sigma$ is MVW-inv. for $\widetilde{Sp}_{2n}$.

⑥ <u>$U(V)$</u>. Let $\{e_1, \cdots, e_n\}$ be an orthogonal basis of V/E, (E/F)

Let $\delta \in GL_F(V)$ be $\delta(\sum a_i e_i) = \sum \bar{a_i} e_i$, $a_i \in E$

Set $\sigma : U(V) \to U(V)$ by $\sigma(g) = \delta g \delta^{-1}$.

<u>Thm</u> (Lin – Sun – Tan, 2014)

MVW-involution does not exist for quaternionic classical groups

Thm (Emory-T, 2023)

MVW-involution exists for $GSpin(V)$ and $GPin(V)$.

- Let V be quadratic space over F w/ $q : V \to F$ the quadratic form.
  There exists a unique a non-trivial GL_1-ext. of $O(V)$ and $SO(V)$, called $GPin(V)$ and $GSpin(V)$, i.e.

$$\begin{array}{ccccccccc} 1 & \to & GL_1 & \to & GPin(V) & \to & O(V) & \to & 1 \\ & & \| & & \cup\!| & & \cup\!| & & \\ 1 & \to & GL_1 & \to & GSpin(V) & \to & SO(V) & \to & 1 \end{array} \quad , \quad GPin(V)^{\circ} = GSpin(V)$$

- More concretely

- $T(V) = \bigoplus_{\ell=0}^{\infty} V^{\otimes \ell}$, where $V^{\otimes \ell} = \underbrace{V \otimes \cdots \otimes V}_{\ell \text{ times}}$, $V^{\otimes 0} = F$

  tensor algebra

- $C(V) = T(V) / \langle v \otimes v - q(v) \mid v \in V \rangle$, $q : V \to F$ quadratic form.

  Clifford algebra

- $C^{\ell}(V) =$ the image of $V^{\otimes \ell}$ in $C(V)$, so each $x \in C^{\ell}(V)$ is written as $x = \sum v_1 v_2 \cdots v_\ell$, $v_i \in V$.

$$C(V) = \sum_{\ell=0}^{\infty} C^{\ell}(V) = \sum_{\ell=0}^{\dim V} C^{\ell}(V)$$

$$C^+(V) = \sum_{\ell=\text{even}} C^{\ell}(V) \quad , \quad C^-(V) = \sum_{\ell=\text{odd}} C^{\ell}(V)$$

so $C(V) = \underbrace{C^+(V)}_{\text{even Clifford algebra}} \oplus C^-(V)$

Def: For $v_1 v_2 \cdots v_\ell \in C^{\ell}(V)$, define
$(v_1 v_2 \cdots v_\ell)^* = v_\ell v_{\ell-1} \cdots v_1$ (canonical involution)
For $x = x^+ + x^- \in C(V)$, $x^+ \in C^+(V)$, $x^- \in C^-(V)$
define $\alpha(x) = x^+ - x^-$, and $\bar{x} = \alpha(x)^* = \alpha(x^*)$

Def: $GPin(V) = \{ x \in C(V)^{\times} \mid \alpha(x) v x^{-1} \in V \text{ for all } v \in V \}$
$GSpin(V) = GPin(V) \cap C^+(V)$.

Prop: For each $x \in GPin(V)$, define $r_x : V \to V$, $r_x(v) = \alpha(x) v x^{-1}$.

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Prop: For each $x \in GPin(V)$, define $r_x : V \to V$, $r_x(v) = \alpha(x) v x^{-1}$.
Then $r_x \in O(V)$ and if $x \in GSpin(V)$, then $r_x \in SO(V)$, so

$$\begin{array}{ccccccccc} 1 & \to & GL_1 & \to & GPin(V) & \xrightarrow{r} & O(V) & \to & 1 \\ & & \| & & \cup\!\mid & & \cup\!\mid & & \\ 1 & \to & GL_1 & \to & GSpin(V) & \to & SO(V) & \to & 1 \end{array}$$

Note: · Both $GPin(V)$ and $GSpin(V)$ are closed under $*$ and α.
· For $g \in GPin(V)$, $g\alpha(g) \in F^\times$, giving $N : GPin(V) \to F^\times$, called Clifford norm. $\mathrm{Ker}(N) = Pin(V)$ or $Spin(V)$.

Def: $\beta : GPin(V) \to GPin(V)$ by

$$\beta(g) = \begin{cases} g^* & \text{if } \dim V = 2k \\ \alpha^{k+1}(g) & \text{if } \dim V = 2k-1 \end{cases}$$

Thm: β^{-1} is an MVW-involution for $GPin(V)$. Also

Define $\sigma : GSpin(V) \to GSpin(V)$

$$\sigma(g) = \begin{cases} \beta(g)^{-1} & \text{if } \dim V = 2k-1 \ (k = \text{any}) \text{ or } 2k \text{ w/ } k = \text{even}. \\ \eta\beta(g)^{-1}\eta^{-1} & \text{if } \dim V = 2k \ (k = \text{odd}), \text{ where } \eta \in GPin(V) \setminus GSpin(V) \end{cases}$$

Then σ is an MVW-involution for $GSpin(V)$

Cor: For $\pi \in \mathrm{Irr}(GPin(V))$

$$\pi^\vee = \begin{cases} \omega_\pi^{-1} \circ N \otimes \pi & (\dim V = 2k-1) \\ (\omega_\pi^{k} \circ \mathrm{sign}) \cdot (\omega_\pi^{-1} \circ N) \otimes \pi & (\dim V = 2k) \end{cases}$$

$N : GPin(V) \to GL_1$

ω_π = central char $|_{Z(GPin(V))^\circ}$

where $\mathrm{sign} : GPin(V) \to \{\pm 1\} = GPin(V)/GSpin(V)$

For $\pi \in \mathrm{Irr}(GSpin(V))$

$$\pi^\vee = \begin{cases} \omega_\pi^{-1} \circ N \otimes \pi & \dim V = 2k-1 \ (k = \text{any}) \text{ or } 2k \ (k = \text{even}) \\ \omega_\pi^{-1} \circ N \otimes \pi^\delta & \text{for } \delta \in GPin(V) \setminus GSpin(V), \ \dim V = 2k \ (k = \text{odd}) \end{cases}$$

Idea of proof

Thm (Harish-Chandra) Given $\pi \in \mathrm{Irr}(G)$ and $f \in C_c^\infty(G)$,

$$\mathrm{trace}(\pi(f)) = \int_G f(g) \chi_\pi(g)\, dg \quad \text{for some } \chi_\pi : G \to \mathbb{C}.$$

Further, $\pi_1 \cong \pi_2$ iff $\chi_{\pi_1} \cong \chi_{\pi_2}$

Cor: $\chi_{\pi^\vee}(g) = \chi_\pi(g^{-1})$

Cor: $\chi_{\pi^\vee}(g) = \chi_\pi(g^{-1})$

e.g.) GL_n : each s.s. g^{-1} is conjugate to ${}^t g^{-1}$

$O(V)$: " " " g

$GPin(V)$: " " " $\beta(g)$

etc.

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③ Application to multiplicity one.

Thm (Emory - T)

Let $W \subseteq V$ be quadratic spaces s.t. $\dim V = \dim W + 1$, so that $GPin(W) \subseteq GPin(V)$ lifting $O(W) \subseteq O(V)$.

For $\pi \in Irr(GPin(V))$, $\tau \in Irr(GPin(W))$.

$\dim_{\mathbb{C}} \underbrace{Hom_{GPin(W)}(\pi, \tau)}_{\text{GGP-period}} \leq 1$. Similarly for GSpin.

Note For $(O(V), O(W))$, (Sp_{2n}, Sp_{2n-1}), $(U(V), U(W))$, (GL_n, GL_{n-1}) proven by Aizenbud - Gourevitch - Rallis - Schiffman.

For $(SO(V), SO(W))$ by Waldspurger.

Archimedean cases by Sun - Zhu.