Algorithms for computing parabolic inductions and Jacquet modules for GL over local fields

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TYMRC
Conference on Real Reductive Groups and Theta Correspondence
25 July, 2024

Parabolic inductions and Jacquet modules

- Let $G_n := \mathrm{GL}_n(F)$, where F is a local field. Let P_{n_1,n_2} be parabolic subgroup of elements of the form:

$$\begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix}, \quad g_1 \in G_{n_1}, g_2 \in G_{n_2}$$

- All representations (over $\mathbb{C}$) are
  - smooth for p-adic F;
  - Casselman-Wallach (smooth Fréchet space of moderate growth) for real F.
- Let $\pi_1 \in \mathrm{Rep}(G_{n_1})$ and $\pi_2 \in \mathrm{Rep}(G_{n_2})$. The normalized parabolically induced module is defined as:

$$\pi_1 \times \pi_2 := \left\{ f : G_n \to \pi_1 \boxtimes \pi_2 : f(pg) = \delta(p)^{1/2} p.(f(g)) \right\}.$$

- Let U_{n_1,n_2} be the unipotent radical of P_{n_1,n_2}. Jacquet modules $r_{n_1,n_2}(\pi)$:
  - (p-adic) $\delta^{-1/2} \cdot \pi / \mathrm{span}\left\{ u.x - x : u \in U, x \in \pi \right\}$;
  - (real) $\delta^{-1/2} \cdot \pi / \overline{(\mathfrak{u}.\pi)}$, where is $\mathfrak{u} = \mathrm{Lie}(U)$.
- p-adic: studies from Tadić school. Real: early work of Speh and Vogan

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Applications of parabolic inductions and Jacquet modules

Many things involve parabolic induction and Jacquet modules:

- Langlands classification (irreducible repns)
- Unitary dual: normalized parabolic induction preserves unitarity
- Theta correspondence
- Branching laws (e.g. Gan-Gross-Prasad problems)
- Aubert-Zelevinsky duality (p-adic)

$$\sum_{P,\mathrm{rank}(P)=i} (-1)^i [\mathrm{Ind}_P^G(r_P(\pi))]$$

- Dirac cohomology

Parabolic induction/Jacquet module as a basic case of branching law

- Let $\pi \in \mathrm{Irr}(G_{n+1})$. Let $\pi' \in \mathrm{Irr}(G_n)$ and let χ be a G_1-character. Then

$$\begin{aligned} & \mathrm{Hom}_{G_{n+1}}(\pi, \pi' \times \chi) \\ \overset{\text{Frobenius reciprocity}}{\cong} \; & \mathrm{Hom}_{P_{n,1}}(\pi, \delta^{1/2} \cdot (\pi' \boxtimes \chi)) \\ \overset{\text{Forget } U_{n,1} \times G_1\text{-action}}{\hookrightarrow} \; & \mathrm{Hom}_{G_n}(\pi, \delta^{1/2} \cdot \pi') \end{aligned}$$

The last space has dimension at most one (Aizenbud-Rallis-Gourevitch-Schiffman, Sun-Zhu), and so the former one also has dimension at most one.

- Goal: Explain some *generalizations for parabolic inductions and Jacquet modules in this theme*, and then explain applications to branching laws

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p-adic group picture: Big derivatives

We first consider F is a p-adic field.

Definition (Big derivatives)

Let $\pi \in \mathrm{Irr}(G_n)$. Let $\sigma \in \mathrm{Irr}^{generic}(G_k)$. Define

$$\mathbb{D}_\sigma(\pi) := \mathrm{Hom}_{G_k}(\sigma, r_{n-k,k}(\pi)),$$

with natural G_{n-k}-action defined by: $g \in G_{n-k}$,

$$(g.f)(x) = \begin{pmatrix} g & \\ & I_k \end{pmatrix} .(f(x))$$

One may consider the above notion is analogue to the *big theta lift*, that one replaces the Weil representation by the Jacquet module of an arbitrary representation.

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Structure of big derivatives

Theorem (C. 24)

Let $\pi \in \mathrm{Irr}(G_n)$ and let σ be a generic representation of G_k. Then $\mathbb{D}_\sigma(\pi)$ (if non-zero) is socle-irreducible i.e. has unique simple submodule and the submodule appears with multiplicity one in the Jordan-Hölder series of $\mathbb{D}_\sigma(\pi)$.

The above result extends result of Jantzen, Mínguez and Lapid-Mínguez.

Definition

Define $D_\sigma(\pi)$ to be the unique simple submodule of $\mathbb{D}_\sigma(\pi)$[1]

[1] Any generic representation σ is a product $\delta_1 \times \ldots \times \delta_r$ of essentially square integrable representations. From this,

$$D_\sigma(\pi) \cong D_{\delta_r} \circ \ldots \circ D_{\delta_1}(\pi).$$

So later on, we focus on derivatives of essentially square integrable repns.

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Derivatives on Langlands parameter

To explain algorithms, we need combinatorial ingredients:

1. Segment is a data $[a,b]_\rho$ with cuspidal repn ρ and $b-a \in \mathbb{Z}$
2. (Zelevinsky) There is a 1-1 correspondence:

$$\text{Segments} \longleftrightarrow \text{Discrete series}$$

denote by

$$\Delta \longleftrightarrow \mathrm{St}(\Delta).$$

3. There are two parametrizations for irreducibles

$$\text{Multisegments} \longleftrightarrow \text{Irreducible representations}$$

(Langlands parameter)

$$\mathfrak{m} \longleftrightarrow L(\mathfrak{m}) := \text{irr. quotient of standard repns}$$

(Zelevinsky parameter)

$$\mathfrak{m} \longleftrightarrow \langle \mathfrak{m} \rangle := \text{Aubert-Zelevinsky dual of } L(\mathfrak{m})$$

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Algorithms on Derivatives

- We shall present two algorithms of our need, which do not need full Mœglin-Waldspurger algorithm, and the complexity of computations is roughly in terms of the (number of segments in $\mathfrak{m}$)^2 $\rightsquigarrow$ much intuitive and applicable
- Most of algorithms involve steps of picking linked segments in neighbors i.e. of the form $[a, b]$ and $[a', b-1]$ with $a' < a$.
    - It could take downward, which means to fix one $[a, b]$ and then find $[a', b-1]$, and usually we choose the 'minimal/shortest' one.
    - It could also take upward, which means to fix $[a', b-1]$ and then find $[a, b]$, and usually we choose the 'maximal/longest' one.

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Derivatives on Zelevinsky parameters

Given $\Delta = [a, b]_\rho$, compute $D_{\mathrm{St}(\Delta)}(\langle \mathfrak{m} \rangle)$.

1. Remove upward maximal sequences of linked segments in neighbors in $\mathfrak{m}$ (multiple times) ranging $a-1$ to b $\rightsquigarrow$ a new multisegment $\mathfrak{m}'$
2. Choose downward minimal sequence of lined segments in neighbors in $\mathfrak{m}'$ from $\nu^b\rho$ to $\nu^a\rho$, denoted by $\Delta_1, \dots, \Delta_r$
3. Define $\mathcal{D}^{Zel}_\Delta(\mathfrak{m}) = \mathfrak{m} - \Delta_1 - \dots - \Delta_r + \Delta_1^- + \dots + \Delta_r^-$, where for $\Delta_i = [a_i, b_i]_\rho$, define

$$\Delta_i^- := [a_i, b_i - 1]_\rho.$$

Theorem (C.-Pattanayak, in preparation)

Let $\pi \in \mathrm{Irr}$. Let $\pi = \langle \mathfrak{m} \rangle$. Then $D_{\mathrm{St}(\Delta)}(\langle \mathfrak{m} \rangle) \cong \langle \mathcal{D}^{Zel}_\Delta(\mathfrak{m}) \rangle$.

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Examples on $\mathcal{D}^{Zel}_{\Delta}$

Example

Let $\mathfrak{m} = \{[0,4],[1,4],[0,5],[2,5],[1,6],[5,6]\}$.have to find $D^{Zel}_{[4,6]}(\mathfrak{m})$. There is nothing for Step 1. For Step 2, we have the sequence $[5,6]$, $[2,5]$, $[1,4]$. Hence

$$\mathcal{D}^{Zel}_{[4,6]}(\mathfrak{m}) = \{[0,4],[1,3],[0,5],[2,4],[1,6],[5]\}.$$

Example

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Examples on $\mathcal{D}_{\Delta}^{Zel}$

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Derivatives on Langlands parameters

Given $\Delta = [a, b]_\rho$, compute $D_{\mathrm{St}(\Delta)}(L(\mathfrak{m}))$. Outline:

1. Choose upward minimal sequences of linked segments in $\mathfrak{m}$ (multiple times)
2. Difference between a pair of linked segments tell the possible points to be 'picked' e.g. the possible points for $[4, 9], [1, 7]$ is 1 and 2.
3. Pick those points (according to Δ) and remove them to get new segments
4. The resulting segment is denoted by $\mathcal{D}^{Lang}_{\Delta}(\mathfrak{m})$

This gives

$$D_{\mathrm{St}(\Delta)}(L(\mathfrak{m})) \cong L(\mathcal{D}^{Lang}_{\Delta}(\mathfrak{m})).$$

Examples on $\mathcal{D}_{\Delta}^{Lang}$

Example

Let $\mathfrak{m} = \{[0,2],[2,4],[1]\}$ and let $\Delta = [0,1]$. Then

1. Step 1 gives two sequences:

   1. $[0,2],[2,4]$
   2. $[1]$

2. The first sequence has 'freedom' on 0 and the second one has 'freedom' on 1.
3. Then $D_{[0,1]}^{Lang}(\mathfrak{m}) = \{[1,2],[2,4]\}$.

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ε-invariant and η-invariant

Definition

Let $\pi \in \mathrm{Irr}$ and let $\sigma \in \mathrm{Irr}^{sq.int.}(G_k)$ (set of essentially square-integrable repns). Define

$$\varepsilon_\sigma(\pi)$$

to be the largest integer such that $D^{\varepsilon_\sigma(\pi)}(\pi) \neq 0$.

Zelevinsky computed the Jacquet functor of σ and showed that

$$r_{i,k-i}(\sigma) \cong \sigma_i \boxtimes \sigma_i'$$

for some $\sigma_i \in \mathrm{Irr}^{sq.int.}$ and $\sigma_i' \in \mathrm{Irr}^{sq.int.}$.

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Representation-theoretic interpretation of η-invariants

We associate $\eta_\sigma(\pi)$ with the following representation:

$$\omega := (\sigma_k)^{\times \varepsilon_{\sigma_k}(\pi)} \times \ldots \times (\sigma_1)^{\times \varepsilon_{\sigma_1}(\pi)}$$

Most of the use or application of η_σ-invariant is based on the following representation-theoretic result on Jacquet modules:

Proposition (C.)

Then

$$D_\omega(\pi) \boxtimes \omega$$

is a direct summand in $r_{p,q}(\pi)$. In other words,

$$\mathbb{D}_\omega(\pi) = D_\omega(\pi)$$

is irreducible.

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Integrals (p-adic)

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Let $\sigma \in \mathrm{Irr}^{sq.int.}$ and let $\pi \in \mathrm{Irr}$. Define $I_\sigma(\pi)$ to be the unique simple submodule such that

$$I_\sigma(\pi) \hookrightarrow \sigma \times \pi.$$

It is now a standard result due to the work of Lapid-Mínguez and C.:

Theorem

$\sigma \times \pi$ is socle-irreducible.

Algorithm for computing $I_\sigma(\pi)$ is a sort of reverse process of the derivative algorithm $\mathcal{D}_\Delta$.

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Notion of (generalized GGP) relevant pairs

Definition

Let $\sigma, \sigma' \in \mathrm{Irr}^{sq.irr}$. Let $\pi \in \mathrm{Irr}$. We say that (σ, σ', π) is a *commutative triple* if $D_\sigma(\pi) \neq 0$ and

$$\eta_\sigma(I_\sigma(\pi)) = \eta_\sigma(\pi).$$

Theorem (C.)

Let (σ, σ', π) *be a commutative triple. Then*

$$D_\sigma \circ I_{\sigma'}(\pi) \cong I_{\sigma'} \circ D_\sigma(\pi).$$

Example

If $\mathrm{csupp}(\sigma) \cap \mathrm{csupp}(\sigma') = \emptyset$, then (σ, σ', π) is always a commutative triple.

Application on branching laws

For branching problems, the most natural one is to consider *sequence* of derivatives in a suitable ordering.

Definition

Let $\sigma_1, \ldots, \sigma_r \in \mathrm{Irr}^{sq.irr}$. We say that $\sigma_1, \ldots, \sigma_r$ are in an *ascending order* if

$$\sigma_1 \times \ldots \times \sigma_r$$

is the dual of a Langlands standard module[2]

[2]Dual of a Langlands standard module has unique simple quotient and that simple quotient is generic. This properties allow one to construct simple quotients of Bernstein-Zelevinsky derivatives of an irreducible representations. This is how such notion appears in branching laws.

Application on branching laws

Definition

Let F be p-adic. Let $\pi \in \mathrm{Irr}(G_n)$ and let $\pi' \in \mathrm{Irr}(G_m)$. Then (π, π') is said to be *relevant* if there exist ascending sequences $\{\sigma_1, \ldots, \sigma_r\}$ and $\{\sigma'_1, \ldots, \sigma'_s\}$ such that

1. For all i, j, $(\sigma_i, \sigma'_j, I_{\sigma'_{j-1}} \circ \ldots \circ I_{\sigma'_1} \circ D_{\sigma_{i-1}} \circ \ldots \circ D_{\sigma_1}(\pi))$ is a commutative triple;
2. $I_{\sigma'_s} \circ \ldots \circ I_{\sigma'_1} \circ D_{\sigma_r} \circ \ldots \circ D_{\sigma_1}(\nu^{1/2} \cdot \pi) \cong \pi'$

Theorem

Let $\pi \in \mathrm{Irr}(G_{n+1})$ and let $\pi' \in \mathrm{Irr}(G_n)$. Then $\mathrm{Hom}_{G_n}(\pi, \pi') \neq 0$ if and only if (π, π') is relevant.

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Construct Schur-Weyl duality

We now switch to $\mathrm{GL}_n(\mathbb{C})$. To transfer to real groups, the tool is the Schur-Weyl type duality constructed with Daniel Wong.

1. Take $G = \mathrm{GL}_n(\mathbb{C})$ and take a $(\mathfrak{g}, K)$-module X of $\mathrm{GL}_n(\mathbb{C})$. Take $V = \mathbb{C}^n$ with G acting by:

$$g.v = g^{-T} v.$$

   Take $K = U(n)$.
2. The space $(X \otimes V^{\otimes m})^K$ has a graded Hecke algebra action $\rightsquigarrow$ gives a $\mathrm{GL}_m(\mathbb{Q}_p)$-repn
3. Many good properties, e.g. parabolic induction and irreducibility, transferred from $\mathrm{GL}_n(\mathbb{C})$-repn to $\mathrm{GL}_m(\mathbb{Q}_p)$-repn

Construct Schur-Weyl duality

We now switch to $\mathrm{GL}_n(\mathbb{C})$. To transfer to real groups, the tool is the Schur-Weyl type duality constructed with Daniel Wong.

1. Take $G = \mathrm{GL}_n(\mathbb{C})$ and take a $(\mathfrak{g}, K)$-module X of $\mathrm{GL}_n(\mathbb{C})$. Take $V = \mathbb{C}^n$ with G acting by:

$$g.v = g^{-T}v.$$

   Take $K = U(n)$.
2. The space $(X \otimes V^{\otimes m})^K$ has a graded Hecke algebra action $\rightsquigarrow$ gives a $\mathrm{GL}_m(\mathbb{Q}_p)$-repn
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$\mathrm{GL}_n(\mathbb{C})$-results via Schur-Weyl type duality

Consequences:

Theorem (C.-Wong)

1. *Let* $\pi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{C}))$ *and let* χ *be a character of* $\mathrm{GL}_1(\mathbb{C})$*. Then* $\pi \times \chi$ *has unique simple quotient and has unique simple submodule. Moreover, there is an algorithm on Langlands parameters to compute the simple quotient and submodule.*
2. *Let* $\pi \in \mathrm{Irr}(\mathrm{GL}_{n+1}(\mathbb{C}))$ *and let* χ *be a character of* $\mathrm{GL}_1(\mathbb{C})$*. Let* $\mathfrak{u}$ *be the Lie algebra of the unipotent radical of* $P_{n,1} \subset G_{n+1}$*. Then* $\pi/\overline{(\mathfrak{u}.\pi)}$ *has at most one simple quotient of the form* $D(\pi) \boxtimes \chi$*. Moreover, there is an algorithm in Langlands parameter to* $D_\chi(\pi)$*.*

Algorithms for computing derivatives for $\mathrm{GL}_n(\mathbb{C})$

Let $\pi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{C}))$ and let χ be a character of $\mathrm{Irr}(\mathrm{GL}_1(\mathbb{C}))$. Compute $D_\chi(\pi)$ i.e.

$$\pi \hookrightarrow D_\chi(\pi) \times \chi,$$

one applies the following diagram:

$\mathrm{GL}_n(\mathbb{C})$ − side $\qquad$ $\mathrm{GL}_m(\mathbb{Q}_p)$ − side

(1): Schur-Weyl type duality to some $\mathrm{GL}(\mathbb{Q}_p)$-representations
(2): Derivative algorithm D_σ [3]
(3): Schur-Weyl type duality back to a $\mathrm{GL}(\mathbb{C})$-representation

[3] Even $D_\sigma(\widetilde{\pi}) \neq 0$, the repn under Schur-Weyl duality may not be a $\mathrm{GL}_{n-1}(\mathbb{C})$. In such case, $D_\chi(\pi)$ does not exist.

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$$\begin{array}{ccc} \pi, \chi & \xrightarrow{(1)} & \widetilde{\pi}, \sigma_\chi \\ & & \downarrow (2) \\ D_\chi(\pi) & \xleftarrow[(3)]{} & D_\sigma(\widetilde{\pi}) \end{array}$$

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Example on $D_\chi(\pi)$ (real)

Some notations

- For $a, b \in \mathbb{C}$ with $b - a \in \mathbb{Z}$, $\chi_{a,b} : \mathbb{C}^\times \to \mathbb{C}^\times$ given by $\chi_{a,b}(z) = z^a \bar{z}^b$
- Then the irreducible modules are the quotients of Langlands standard modules of the form

$$\chi_{a_1,b_1} \times \chi_{a_2,b_2} \times \ldots \times \chi_{a_r,b_r}$$

with $\mathrm{Re}(a_1 + b_1) \geq \mathrm{Re}(a_2 + b_2) \geq \ldots \geq \mathrm{Re}(a_r + b_r)$.

Example on $D_\chi(\pi)$ (real)

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We consider the Langlands quotient π of the standard module

$$\chi_{6.5,3.5} \times \chi_{4.5,1.5} \times \chi_{4.5,-0.5} \times \chi_{1.5,0.5}.$$

One computes $D_{\chi_{3.5,-0.5}}(\pi)$.

1. Under Schur-Weyl type duality, π corresponds to $L([4,6],[2,4],[1],[0,4])$ and χ corresponds to $\mathrm{St}([0,3])$.
2. Since $\mathcal{D}^{Lang}_{[0,3]}(\{[4,6],[2,4],[1],[0,4]\}) = \{[2,4],[4,6],[1],[4]\}$ contains more than three segments, $D_{\chi_{3.5,-0.5}}(\pi)$ does not exist.

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Remarks on Algorithms on Computing $\pi/(\overline{\mathfrak{u}.\pi})$

- Determine explicit quotient branching law for basic cases
- Determine Archimedean highest Bernstein-Zelevinsky derivative (defined by Gomez-Gourevitch-Sahi) of an arbitrary representation for $\mathrm{GL}_n(\mathbb{C})$
  $\rightsquigarrow$ use the Jacquet functor relation between $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{Q}_p)$ representations to transfer to a problem of $\mathrm{GL}_m(\mathbb{Q}_p)$
- One expects similar results for $\mathrm{GL}_n(\mathbb{R})$ using Ciubotaru-Trapa construction of Schur-Weyl duality

Thickened subcategory

Actually, for some applications e.g. local non-tempered GGP, we need some higher structure (beyond simple quotients).

Definition

1. A multisegment $\mathfrak{m} = \{[a_1, b_1]_1, \ldots, [a_r, b_r]_1\}$ is said to be *thickened* if $b_j > a_i$ for all i, j.
2. An irreducible representation π of $GL(\mathbb{Q}_p)$ is said to be *thickened* if $\pi \cong \langle \mathfrak{m} \rangle$ for some thickend multisegment $\mathfrak{m}$.
3. A Serre subcategory of the $GL(\mathbb{Q}_p)$-representation category is said to be *thickened* if all its representations are of finite length and all simple composition factors are thickened.

Equivalence of categories

Theorem (In progress)

Let $\mathcal{HC}_\lambda$ *be the Serre subcategory of* $\mathrm{GL}_n(\mathbb{C})$ *Harish-Chandra category of finite length objects with simple composition factors of a fixed infinitesimal character* λ*. Then*

1. *there exists a thickened subcategory* $\mathcal{TH}$ *of some* $\mathrm{GL}_m(\mathbb{Q}_p)$*-representation category such that the two categories*

$$\mathcal{HC}_\lambda \longleftrightarrow \mathcal{TH}$$

are equivalent;

2. *parabolic induction and Jacquet functor behave well under the equivalence of categories.*

Indeed, the category $\mathcal{TH}$ can be described explicitly. Upshot: parabolic induction and Jacquet functor could be easier via more established geometric lemma and exactness of Jacquet functors.

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One may expect for branching laws beyond basic case?

1. As in the p-adic case, one expects to need a sequence of derivatives for real groups.
2. The tensor product problem also appears. We give an example from the equal rank Fourier-Jacobi model:

$$\pi \widehat{\otimes} \mathcal{S}(\mathbb{C}^n)$$

which admits a short exact sequence from Borel's lemma:

$$0 \to \pi \widehat{\otimes} \mathcal{S}(\mathbb{C}^n - 0) \to \pi \widehat{\otimes} \mathcal{S}(\mathbb{C}^n) \to \pi \widehat{\otimes} \mathbb{C}[[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]] \to 0.$$

The latter space $\pi \widehat{\otimes} \mathbb{C}[[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]]$ admits a filtration with subquotients of the form:

$$\pi \otimes \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^j(\bar{V}).$$

Via the Schur-Weyl duality, the structure can be computed from the Bernstein-Zelevinsky derivatives from $\mathrm{GL}_m(\mathbb{Q}_p)$ (with inputs of result joint with Savin).

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Happy Birthday, Chengbo!