Casselman's comparison conjecture

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$(\mathfrak{g}, K)$ modules

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$(\mathfrak{g}, K)$ modules

- Let G be a linear reductive real Lie group with a maximal compact subgroup K. Write $\mathfrak{g}$ and $\mathfrak{k}$ for the complexified Lie algebras of G and K, respectively. Let θ be the Cartan involution of G with $G^\theta = K$.
- A $(\mathfrak{g}, K)$ *module* (π, W) is a complex linear space which carries compatible actions of $\mathfrak{g}$ and K in the sense that
  (1) the differential of the K action is equal to the $\mathfrak{k}$ $(\subset \mathfrak{g})$ action;
  (2) $\pi(\mathrm{Ad}(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$ $(\forall (k, X) \in K \times \mathfrak{g})$;
  (3) $\dim_{\mathbb{C}} \mathrm{span}\{\pi(k)v : k \in K\} < \infty$ $(\forall v \in W)$.

Harish-Chandra modules

- A $(\mathfrak{g}, K)$ module W is said to be admissible if for any finite-dimensional complex linear irreducible representation σ of K, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_K(\sigma, W) < \infty.$$

- A $(\mathfrak{g}, K)$ module W is said to be finitely generated if there exists a finite-dimensional subspace $F \subset W$ such that

$$W = \mathcal{U}(\mathfrak{g})F,$$

where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

Harish-Chandra modules

- A $(\mathfrak{g}, K)$ module is called a *Harish-Chandra module* if it is finitely generated and admissible.
- If U is an irreducible unitary representation of G, then by a theorem of Harish-Chandra

$$U_K := \{v \in U : \dim_{\mathbb{C}} \operatorname{span}\{\pi(k)v : k \in K\} < \infty\}$$

  is a Harish-Chandra module.
- Write $\mathcal{H}(\mathfrak{g}, K)$ for the category of Harish-Chandra modules, which is an abelian category, called the *Harish-Chandra category*.

Globalization

- Harish-Chandra modules are convenient for studying algebraic properties of representations. However, for application in the automorphic form theory and harmonic analysis, etc, we also need to study continuous representations of G.
- Let $W \in \mathcal{H}(\mathfrak{g}, K)$. Schmid constructed a *minimal globalization* W_{min} which consists of analytic vectors. Casselman and Wallach constructed a *smooth globalization* W_∞ which is a moderate growth smooth Fréchet representation of G. They fit into a sequence

$$W \subset W_{\text{min}} \subset W_\infty.$$

Cassleman's comparison conjecture

- It is a natural question to compare homological properties of a Harish-Chandra module and its globalization.
- Let P be a real standard parabolic subgroup of G with a Levi decomposition $P = LN$ where $\theta(L) = L$. Casselman's comparison conjecture says that

$$H_*(\mathfrak{n}, W_\infty) \cong (H_*(\mathfrak{n}, W))_\infty.$$

Cassleman's comparison conjecture

- There is an analogous conjecture for the minimal globalization.
- For minimal globalization, the comparison conjecture was shown by Hecht-Taylor (1993), Bunke-Olbrich (1997), Bratten (1998) in different generalities. Their proofs are all based the theory of analytic D-modules.

Cassleman's comparison conjecture

- When P a real minimal standard parabolic subgroup, Hecht-Taylor (1995) deduced the comparison conjecture from its minimal globalization analogue.
- We give a proof of Casselman's comparison conjecture for smooth globalization while P is a minimal real standard parabolic subgroup using the distribution theory alone.

Idea of proof

First, by a theorem of Casselman-Osborne

$$\dim H_*(\mathfrak{n}, W) = \dim H^*(\mathfrak{n}, W^*) < \infty.$$

Then, the homological comparison W between W_∞ follows from the cohomological comparison W^* between $(W_\infty)'$.

Idea of proof

Second, by a reduction due to Hecht-Taylor, the cohomological comparison reduces to the case that

$$W_\infty = I(\sigma) := \mathrm{Ind}_P^G(\sigma \otimes \mathbf{1}_N)$$

is a principal series.

Idea of proof

Put

$$X = G/P$$

and

$$E(\sigma) = G \times_P \sigma.$$

Then, W_∞ is equal to the space of smooth sections of $E(\sigma)$ and $(W_\infty)'$ is equal to the space of tempered $E(\sigma)$-distributions on X.

Bruhat filtration

The action of N on $X = G/P$ gives a stratification

$$\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_r = X$$

with $r = |W|$ (W = Weyl group) and each

$$Z_k - Z_{k-1} = Nw_k P/P = C(w_k), \quad w_k \in W$$

is a Bruhat cell, which is a locally closed algebraic subvariety of X.

Bruhat filtration

Write $C_k^\infty(\sigma)$ for the space of smooth sections of $E(\sigma)$ which vanish with all derivatives along Z_k. Then, we have a $\mathcal{U}(\mathfrak{g})$ module filtration:

$$I(\sigma) = C_0^\infty(\sigma) \supset C_1^\infty(\sigma) \supset \cdots \supset C_r^\infty(\sigma) = 0,$$

which is called the Bruhat filtration of $I(\sigma)$.

Bruhat filtration

Put

$$I_{k,\sigma} = C_k^\infty(\sigma)^\perp \subset I(\sigma)'.$$

By duality, we get short exact sequences

$$0 \to I_{k-1,\sigma} \to I_{k,\sigma} \to \mathcal{T}(C(w_k), E(\sigma)) \to 0.$$

Tempered $E(\sigma)$-distributions

- For each $w \in W$, $\mathcal{T}(C(w), E(\sigma))$ is called the space of tempered $E(\sigma)$-distributions on X supported in $C(w)$.
- For each $p \in \mathbb{Z}_{\geq 0}$, write $F_p\mathcal{T}(C(w), E(\sigma))$ for the subspace of distributions in $\mathcal{T}(C(w), E(\sigma))$ of transversal degree $\leq p$ and write

$$\begin{aligned} & \mathrm{Gr}^p\, \mathcal{T}(C(w), E(\sigma)) \\ = \; & F_p\mathcal{T}(C(w), E(\sigma))/F_{p-1}\mathcal{T}(C(w), E(\sigma)). \end{aligned}$$

Tempered $E(\sigma)$-distributions

- When $p = 0$,

$$F_0\mathcal{T}(C(w), E(\sigma)) = \mathcal{S}(C(w), E(\sigma))'$$

  is the space of tempered $E(\sigma)$-distributions on $C(w)$.
- For each $p \geq 0$, $\mathrm{Gr}^p\,\mathcal{T}(C(w), E(\sigma))$ admits a P stable finite increasing filtration such that each graded piece is isomorphic to $\mathcal{S}(C(w), E(\sigma))'$ as an $\mathcal{U}(\mathfrak{n})$ module.

Casselman-Jacquet functor

- For a $\mathcal{U}(\mathfrak{p})$ module V, put

$$V^{[\mathfrak{n}]} = \{v \in V : \exists k \geq 0, \mathfrak{n}^k \cdot v = 0\}$$

  and call it the Casselman-Jacquet module of V, which is still a $\mathcal{U}(\mathfrak{p})$ module.
- The Casselman-Jacquet functor is clearly left exact.

Exactness result 1

For any $w \in W$ and any $p \geq 1$, there is an exact sequence

$$\begin{aligned} & 0 \\ \to \quad & (F_{p-1}\mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}]} \\ \to \quad & (F_p\mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}]} \\ \to \quad & (\mathrm{Gr}^p\,\mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}]} \\ \to \quad & 0. \end{aligned}$$

Exactness result 2

For any $k \geq 1$, there is an exact sequence

$$0 \to I_{k-1,\sigma}^{[\mathfrak{n}]} \to I_{k,\sigma}^{[\mathfrak{n}]} \to (I_{k,\sigma}/I_{k-1,\sigma})^{[\mathfrak{n}]} \to 0.$$

Exactness result 3

For any short exact sequence

$$0 \to \sigma_1 \to \sigma_2 \to \sigma_3 \to 0$$

of finite-dimensional representations of P, we show that there is a short exact sequence

$$0 \to I(\sigma_3)'^{[\mathfrak{n}]} \to I(\sigma_2)'^{[\mathfrak{n}]} \to I(\sigma_1)'^{[\mathfrak{n}]} \to 0.$$

The proof

- We need to show that the inclusion $V_K \hookrightarrow V$ induces isomorphisms

$$H^*(\mathfrak{n}, V') = H^*(\mathfrak{n}, V_K^*)$$

  for $V = I(\sigma)$ a principal series.
- By a theorem of Hecht-Schmid, we have

$$H^*(\mathfrak{n}, (V_K^*)^{[\mathfrak{n}]}) = H^*(\mathfrak{n}, V_K^*).$$

The proof

- By Casselman's automatic continuity theorem, we have

$$V'^{[\mathfrak{n}]} = (V_K^*)^{[\mathfrak{n}]}.$$

- Combining the above, it suffices to show that

$$H^*(\mathfrak{n}, I(\sigma)'^{[\mathfrak{n}]}) = H^*(\mathfrak{n}, I(\sigma)').$$

The proof

With various exact sequences as above, it reduces to show that: for any $w \in W$, we have

$$H^*(\mathfrak{n}, \mathcal{S}(C(w), E(\sigma))'^{[\mathfrak{n}]}) = H^*(\mathfrak{n}, \mathcal{S}(C(w), E(\sigma))').$$

The proof

We prove a general comparison statement: for any linear closed subgroup U of N, we have

$$H^*(\mathfrak{n}, \mathcal{S}(N/U)'^{[\mathfrak{n}]}) = H^*(\mathfrak{n}, \mathcal{S}(N/U))').$$

It finishes the proof.

General comparison conjecture

Casselman's comparison conjecture in the general setting is more difficult for various reasons:

(1) the generalization of Hecht-Schmid's theorem and Casselman's automatic continuity theorem fail;

(2) the generalization of the above Bruhat filtration is much more complicated;

(3) in the general setting the conjecture gives no equalities, but says that the maps $H_*(\mathfrak{n}, W) \to H_*(\mathfrak{n}, W_\infty)$ are injective with dense images.

Happy Birthday, Chengbo!