Decomposing tensor product by Dirac cohomology

Jing-Song Huang

Mathematics Research Center
The Chinese University of Hong Kong, Shenzhen

Conference on Real Reductive Groups and Theta Correspondence

Kunmin, China

July 22-26, 2024

Outline

- Kostant algebra and infinitesimal characters in tensor product
- Dirac cohomology and Dirac index
- Decomposing tensor product by Dirac cohomology
- Mehdi-Pandžić-Vogan translation principle

Kostant's Theorems

$\mathfrak{g}$: a complex semisimple Lie algebra.

$U(\mathfrak{g})$: the universal enveloping algebra of $\mathfrak{g}$.

$\mathfrak{h}$: a Cartan subalgebra.

F: a finite-dimensional simple $\mathfrak{g}$-module.

$\Delta(F)$: the multiset of all weights of F.

$$\pi : U(\mathfrak{g}) \to \operatorname{End} X$$

a $\mathfrak{g}$-module with an infinitesimal character χ_λ $(\lambda \in \mathfrak{h}^*)$.

Theorem (Kostant) An infinitesimal character which occurs in the tensor product $X \otimes F$ must be

$\chi_{\lambda+\mu_i}$ for some μ_i in $\Delta(F)$.

Question

Question Which $\chi_{\lambda+\mu_i}$ does occur?

E.g. If $X = \mathbb{C}$ (infl char χ_ρ) and $F = F_\mu$ simple module with highest weight μ, then $\chi_{\rho+\mu_i}$ occurs for $\mu_i = \mu$ only.

A quick answer

F has distinct weights $\mu_1, \ldots, \mu_k$, with multiplicity $d_1, \ldots, d_k$.

Assume that X is a Harish-Chandra module with infinitesimal character χ_λ.

Suppose that characters $\chi_{\lambda+\mu_1}, \ldots \chi_{\lambda+\mu_k}$ are distinct. Then

$$X \otimes F = X_1 \oplus \cdots \oplus X_k$$

such that $X_1, \ldots, X_k$ are the maximal submodules with infinitesimal characters $\chi_{\lambda+\mu_1}, \ldots \chi_{\lambda+\mu_k}$.

Theorem Assume G is of equal rank and χ_λ is regular. Then $H_D(X) \neq 0$ implies that $X_i \neq 0$, provided that $\chi_{\lambda+\mu_i}$ is regular.

The Kostant algebra

F_μ: a simple $\mathfrak{g}$-module with highest weight μ

$$\pi\colon U(\mathfrak{g}) \to \operatorname{End} X$$

be an arbitrary $\mathfrak{g}$-module having an infinitesimal character χ_λ. Kostant introduced the following algebras

$$R = (U(\mathfrak{g}) \otimes \operatorname{End} F_\mu)^{\mathfrak{g}}$$

and

$$R_\pi = (\pi[U(\mathfrak{g})] \otimes \operatorname{End} F_\mu)^{\mathfrak{g}}.$$

There is a surjective homomorphism of algebras $R \to R_\pi$.

Structure of R

Consider the map

$$\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathrm{End}\, F_\mu$$

defined by

$$\delta(x) = x \otimes 1 + 1 \otimes \pi_\mu(x) \text{ for } x \in \mathfrak{g},$$

which extends to a homomorphism of associative algebras.

Then R is commutants of $\delta(U(\mathfrak{g}))$ in $U(\mathfrak{g}) \otimes \mathrm{End}\, F_\mu$.
$\forall u \in Z(\mathfrak{g})$, $\delta(u)$ is in R. Thus, $\delta(Z(\mathfrak{g}))$ is in the center of R.

There is a $\mathfrak{g}$-submodule E of $U(\mathfrak{g})$ such that the multiplication

$$Z(\mathfrak{g}) \otimes E \to U(\mathfrak{g})$$

is a $\mathfrak{g}$-isomorphism. Thus, R is a free $Z(\mathfrak{g})$-module.

Kostant Theorems

Kostant showed that R is a free $Z(\mathfrak{g})$-module of rank equal to

$$r = \sum_{i=1}^{k} d_i^2.$$

Assume that $\pi = \pi_\nu$ is an irreducible finite-dimensional $\mathfrak{g}$-module $F_\pi = F_\nu$ with highest weight ν.

Kostant defined μ to be totally subordinate to ν if the number of irreducible constituents in $F_\nu \otimes F_\mu$ is equal to $d_\mu := \dim F_\mu$. In this case, there is an isomorphism of algebras

$$R_{\pi_\nu} \to \bigoplus_{i=1}^{k} \mathrm{Mat}_{d_i}(\mathbb{C}).$$

Structure of R_π

Theorem (Kostant) Let $u \in Z(\mathfrak{g})$ and $\tilde{u} = (\pi \otimes \pi_\mu)(u)$. Then the operator $\tilde{u}$ in $X \otimes F_\mu$ satisfies the equation

$$\prod_{i=1}^{k}(\tilde{u} - \chi_{\lambda+\mu_i}(u)) = 0.$$

Remark Suppose among $\chi_{\lambda+\mu_1}, \dots \chi_{\lambda+\mu_k}$, m of them that are different. Thus, for any $1 \leq i \leq k$, one has $\lambda + \mu_i$ is $W_\mathfrak{g}$ conjugate to $\lambda + \mu_j$ for some unique j, where $1 \leq j \leq m$. Let k_j be the multiplicity of $\chi_{\lambda+\mu_j}$ so that $\sum k_j = k$. Then $\tilde{u}$ satisfies the equation

$$\prod_{j=1}^{m}(\tilde{u} - \chi_{\lambda+\mu_j}(u))^{k_j} = 0.$$

Minimal polynomial of $\tilde{u}$

Theorem Let X be $\mathfrak{g}$-module with an infinitesimal character χ_λ. Let $u \in Z(\mathfrak{g})$ and $\tilde{u} = (\pi \otimes \pi_\mu)(u)$. Suppose

$$1, \tilde{u}, \tilde{u}^2, \ldots, \tilde{u}^{k-1}$$

are linearly independent in R_π, i.e., the minimal polynomial of $\tilde{u}$ has degree k. Then all generalized infinitesimal characters $\chi_{\lambda+\mu_1}, \ldots, \chi_{\lambda+\mu_m}$ occur in $X \otimes F_\mu$.

Dirac cohomology

G: is a connected semisimple Lie group with finite center.
K: a maximal compact subgroup, with a maximal torus T.
Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the complexified Cartan involution.
B: a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$.
$C(\mathfrak{p})$: the Clifford algebra of $\mathfrak{p}$ with respect to B.

$$D: = \sum_{i=1}^{n} Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p});$$

here $Z_1, \ldots, Z_n$ is an orthonormal basis of $\mathfrak{p}$ with respect to B.
D is independent of the choice of bases and invariant under the diagonal adjoint action of K.
Then

$$D^2 = -C_{\mathfrak{g}} \otimes 1 + C_{\mathfrak{k}_\Delta} + (||\rho_c||^2 - ||\rho||^2) 1 \otimes 1,$$

where ρ and ρ_c are half sums of positive roots for $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$ respectively.

Dirac cohomology

Let X be a $(\mathfrak{g}, K)$-module of finite length.

Definition The Dirac cohomology is defined as follows:

$$H_D(X) \colon = \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D.$$

Then $H_D(X)$ is a finite-dimensional module for the spin double cover $\widetilde{K}$ of K.

Theorem (Huang-Pandžíc) Let X be a $(\mathfrak{g}, K)$-module with an infinitesimal character χ_Λ corresponding to parameter $\Lambda \in \mathfrak{h}^*$. Suppose that $H_D(X)$ contains a representation of $\widetilde{K}$ with infinitesimal character λ. Then Λ and $\lambda \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$ are conjugate under $W_\mathfrak{g}$.

Dirac index

Now assume that G has a compact Cartan subgroup T.

Since $\mathfrak{p}$ is even-dimensional, the spin module S decomposes as $S^+ \oplus \; S^-$, with the $\mathfrak{k}$-submodules $S^\pm$ being the even respectively odd part of S.
Let $X = X_\pi$ be the Harish-Chandra module of an admissible representation π of G or a $(\mathfrak{g}, K)$-module of finite length.

Definition The spin index $I(X)$ is defined to be the following virtual $\widetilde{K}$-module:

$$I(X) = X \otimes S^+ - X \otimes S^-.$$

Dirac index

The Dirac operator D induces the action of the following $\widetilde{K}$-equivariant operators

$$D^{\pm} : X \otimes S^{\pm} \to X \otimes S^{\mp}.$$

D^2 acts by a scalar on each $\widetilde{K}$-type, most of $\widetilde{K}$-modules in $X \otimes S^+$ are the same as in $X \otimes S^-$.
$I(X)$ is a virtual $\widetilde{K}$-module, an integer combination of finitely many $\widetilde{K}$-modules.
Lemma The spin index is equal to the Euler characteristic of Dirac cohomology, i.e.,

$$I(X) = H_D^+(X) - H_D^-(X).$$

Dirac cohomology and Dirac index

Theorem Suppose that an irreducible Harish-Chandra module X has regular infinitesimal character. Then

$$\mathrm{Hom}_{\widetilde{K}}(H_D^+(X), H_D^-(X)) = 0.$$

Thus, Dirac index $I(X) = 0$ iff Dirac cohomology $H_D(X) = 0$.

Decomposing tensor product

Let X be a Harish-Chandra module with a regular χ_λ. Assume that $X \otimes F$ is decomposed into a direct sum of Harish-Chandra modules

$$X \otimes F = X_1 \oplus \cdots \oplus X_k,$$

such that $X_1, \ldots, X_k$ have (distinct) infinitesimal character $\chi_{\lambda+\mu_1}, \ldots, \chi_{\lambda+\mu_k}$ respectively.

Set $\widetilde{E}_\lambda = \mathrm{sign}(w) E_{w\lambda}$ to be a family of finite-dimensional virtual $\widetilde{K}$-module if there is a $w \in W_{\mathfrak{g}}$ such that $w\lambda - \rho_c$ is the highest weight of $\widetilde{K}$-module $E_{w\lambda}$, and $\widetilde{E}_\lambda = 0$ if there is no such a w.

Extend the definition of $\widetilde{E}_\lambda$ to all $\lambda \in \mathfrak{t}^*$.

By definition, $\widetilde{E}$ is a "coherent family".

Main Theorem

Theorem Suppose that

$$I(X) = \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w\lambda},$$

with integers a_w. Then

$$I(X_i) = d_i \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w(\lambda+\mu_i)},$$

with the same integers a_w.

Proof

$$\sum_{i=1}^{k} I(X_i) = I(X) \otimes F = \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w\lambda} \otimes F.$$

Since $\widetilde{E}$ is a coherent family for $\widetilde{K}$, and F can be viewed as a finite-dimensional $\widetilde{K}$-module, one has

$$\sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w\lambda} \otimes F = \sum_{w \in W_{\mathfrak{g}}} a_w \sum_{\mu \in \Delta(F)} \widetilde{E}_{w\lambda+\mu}.$$

Proof

Fix $w \in W_{\mathfrak{g}}$, w permutes $\Delta(F)$.

$$\sum_{\mu \in \Delta(F)} \widetilde{E}_{w\lambda+\mu} = \sum_{\mu \in \Delta(F)} \widetilde{E}_{w\lambda+w\mu}.$$

It follows that

$$\sum_{i=1}^{k} I(X_i) = \sum_{w \in W_{\mathfrak{g}}} a_w \sum_{\mu \in \Delta(F)} \widetilde{E}_{w(\lambda+\mu)} = \sum_{i} d_i \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w(\lambda+\mu_i)}.$$

Separating terms corresponding to different infinitesimal characters

$$I(X_i) = d_i \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w(\lambda+\mu_i)}.$$

Criterion

Corollary Suppose that G is of equal rank and λ is regular.
Then $H_D(X) \neq 0 \Longrightarrow X_i \neq 0$ provided that $\lambda + \mu_i$ is regular.

General case

Suppose among $\chi_{\lambda+\mu_1}, \dots \chi_{\lambda+\mu_k}$, m of them that are different. Thus, for any $1 \leq i \leq k$, one has $\lambda + \mu_i$ is $W_{\mathfrak{g}}$ conjugate to $\lambda + \mu_j$ for some unique j, where $1 \leq j \leq m$.
In this case, Kostant showed that $X \otimes F$ is decomposed into a direct sum of Harish-Chandra modules

$$X \otimes F = Y_1 \oplus \cdots \oplus Y_m,$$

Y_j is the max'l submodule with generalized infl char $\chi_{\lambda+\mu_j}$.

Remark $Z(\mathfrak{g}) = \mathbb{C}[u_1, \dots, u_n]$. Let N be n-dimensional subspace spanned by the u_i. Then $u \to \chi_{\lambda+\mu_j}(u)$ defines a functional L_j on N. $L_i \neq L_j$ $(i \neq j)$. Let N_0 be the Zariski open subset on which $\prod_{i<j}(L_i - L_j) \neq 0$. Fix $u \in N_0$, $\chi_{\lambda+\mu_j}(u)$ $j = 1, \dots, m$ are different. Then $Y_j = \{y \in X \otimes F | (u - \chi_{\lambda+\nu_j}(u))^{k_j} y = 0\}$ is the max'l submodule with generalized infl char $\chi_{\lambda+\mu_j}$.

General case

Theorem Suppose that

$$I(X_\lambda) = \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w\lambda},$$

with the integer coefficients a_w. Then

$$I(Y_j) = \sum_i d_i \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w(\lambda+\mu_i)}, \tag{5.2}$$

with the same integers a_w, and the sum over i is for all i such that $\chi_{\lambda+\mu_i} = \chi_{\lambda+\mu_j}$.

Coherent family

Assume that G has a compact Cartan subgroup T.
Denote by $\Lambda \subset \widehat{T} \subset \mathfrak{t}^*$, the lattice of weights of finite-dimensional representations of G.
Given $\lambda_0 \in \mathfrak{t}^*$, a family of virtual $(\mathfrak{g}, K)$-modules X_λ, $\lambda \in \lambda_0 + \Lambda$, is called coherent if

(i) X_λ has infinitesimal character λ; and

(ii) for any finite-dimensional $(\mathfrak{g}, K)$-module F, and for any $\lambda \in \lambda_0 + \Lambda$,

$$X_\lambda \otimes F = \sum_{\mu \in \Delta(F)} X_{\lambda+\mu}.$$

Coherent family

A virtual $(\mathfrak{g}, K)$-module X with regular $\lambda_0 \in \mathfrak{t}^*$ can be placed in a unique coherent family as above. Then one has

$$I(X_\lambda) \otimes F = \sum_{\mu \in \Delta(F)} I(X_{\lambda+\mu}).$$

This shows that the family $\{I(X_\lambda)\}$ $(\lambda \in \lambda_0 + \Lambda)$ of virtual $\widetilde{K}$-modules has some coherent properties.
It is not a coherent family for $\widetilde{K}$, as $I(X_\lambda)$ does not have $\mathfrak{k}$-infinitesimal character λ.

Translation principle

Theorem (Mehdi-Pandžić-Vogan)
Suppose λ_0 is regular for $\mathfrak{g}$. Let X_λ, $\lambda \in \lambda_0 + \Lambda$, be a coherent family of virtual $(\mathfrak{g}, K)$-modules based on $\lambda_0 + \Lambda$.
Suppose that

$$I(X_{\lambda_0}) = \sum_{w \in W_\mathfrak{g}} a_w \widetilde{E}_{w\lambda_0}.$$

with integer coefficients a_w. Then for any $\mu \in \Lambda$,

$$I(X_{\lambda_0+\mu}) = \sum_{w \in W_\mathfrak{g}} a_w \widetilde{E}_{w(\lambda_0+\mu)},$$

with the same coefficients a_w.

Comparison with coherent family

Proposition

Let X be a Harish-Chandra module with regular infinitesimal character λ_0.

Let X_λ, $\lambda \in \lambda_0 + \Lambda$ be the unique coherent family of virtual $(\mathfrak{g}, K)$-modules with $X_{\lambda_0} = X$.

Assume that $X \otimes F$ is decomposed into a direct sum of Harish-Chandra modules

$$X \otimes F = X_1 \oplus \cdots \oplus X_k,$$

with distinct infinitesimal characters $\lambda + \mu_i$. Then for any $i = 1, \ldots, k$,

$$I(X_i) = d_i I(X_{\lambda_0 + \mu_i}).$$

Comparison with coherent family

Proposition

Let X be a Harish-Chandra module with regular infinitesimal character λ_0.

Let X_λ, $\lambda \in \lambda_0 + \Lambda$ be the unique coherent family of virtual $(\mathfrak{g}, K)$-modules with $X_{\lambda_0} = X$.

Let Y_j be the maximal submodule in $X \otimes F$ with generalised infinitesimal character $\chi_{\lambda_0+\mu_j}$. Then for any $j = 1, \ldots, m$,

$$I(Y_j) = \sum_i d_i I(X_{\lambda_0+\mu_i}),$$

with sum over i such that $\chi_{\lambda_0+\mu_i} = \chi_{\lambda_0+\mu_j}$.

Thank You!

Appendix: twisted Dirac index

Barbasch-Trapa-Pandžić had defined twisted Dirac index $\mathrm{I}_\theta(X)$, which is a virtual $\widetilde{K}$-module, an integer combination of finitely many $\widetilde{K}$-modules as follows

$$\mathrm{I}_\theta(X) = (X \otimes S)^+ - (X \otimes S)^-.$$

Lemma The spin index is equal to

$$\mathrm{I}_\theta(X) = (H_D(X))^+ - (H_D(X))^-.$$

Work in Progress We extend the main results to the non-equal rank case in terms of twisted Dirac index.