Mixed Hodge modules and unipotent representations

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Unitarity and unipotent representations

Let $G_{\mathbb{R}}$ be a real reductive Lie group, i.e., a finite covering of a real form of a connected complex reductive group G.

Classical open problem

Determine the set of irreducible unitary representations of $G_{\mathbb{R}}$.

Proposed shape of a solution (Vogan, Barbasch-Vogan 1980s)

1. Define a set of building blocks, called *unipotent representations*.
2. Show that all unipotent representations are unitary.
3. Show that all unitary representations are built from unipotent ones by induction, cohomological induction and complementary series.

Motivation for unipotent representations

There are several motivations for this idea, e.g., representations of finite groups of Lie type, automorphic forms. I want to focus on:

The orbit method philosophy (Kirillov, . . .)

$$\left\{\begin{matrix}\text{Co-adjoint orbits} \\ \text{of } G_{\mathbb{R}} \circlearrowright \mathfrak{g}_{\mathbb{R}}^* \end{matrix}\right\} \xrightarrow{\text{quantisation}} \left\{\begin{matrix}\text{Unitary representations} \\ \text{of } G_{\mathbb{R}} \end{matrix}\right\}$$

Any element $x \in \mathfrak{g}_{\mathbb{R}}^*$ has a semi-simple part x_s and a commuting nilpotent part x_n.

$$\Downarrow \text{quantise}$$

Any unitary representation is "induced" from a quantisation of a *nilpotent* orbit tensored with a unitary character.

Upshot

Unipotent representations = quantisations of *nilpotent* orbits

Definition of unipotent representation?

The notion of unipotent representation has proved very hard to make precise. It is still unclear what the definition should be.

Stepping stone

What should the *annihilators* of the unipotent representations be?

Recall that admissible representations $\mathbb{V}$ of $G_{\mathbb{R}}$ up to infinitesimal equivalence are in bijection with admissible Harish-Chandra $(\mathfrak{g}, K)$-modules V, where

$$\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}, \quad K = K_{\mathbb{R}} \otimes \mathbb{C},$$

$$K_{\mathbb{R}} \subset G_{\mathbb{R}} \text{ a maximal compact subgroup.}$$

So every irreducible representation $\mathbb{V}$ defines a 2-sided ideal $\mathrm{Ann}(V) \subset U(\mathfrak{g})$.

Definition of unipotent ideals

Construction (Losev-Mason-Brown-Matvieievskyi, following Vogan, Dixmier, Duflo, ...):

- Take $\mathbb{O} \subset \mathfrak{g}^*$ a nilpotent co-adjoint orbit for a complex reductive group G such that $G_{\mathbb{R}}$ covers a real form of G.
- Let $\tilde{\mathbb{O}} \to \mathbb{O}$ be a connected, finite G-equivariant cover.

Theorem (Losev)

There exists an explicitly parametrized space of filtered quantisations $\mathcal{A}_\mu(\tilde{\mathbb{O}})$ of the algebra of functions $\mathbb{C}[\tilde{\mathbb{O}}]$, equipped with homomorphisms $U(\mathfrak{g}) \to \mathcal{A}_\mu(\tilde{\mathbb{O}})$. The space has a natural base point $\mu = 0$, the *canonical quantisation*.

Definition (Losev-Mason-Brown-Matvieievskyi)

The *unipotent ideal* attached to $\tilde{\mathbb{O}}$ is the two-sided ideal

$$I(\tilde{\mathbb{O}}) = \ker(U(\mathfrak{g}) \to \mathcal{A}_0(\tilde{\mathbb{O}})).$$

Unipotent ideals and unitarity

Naive definition

A *unipotent representation of* $G_{\mathbb{R}}$ is an irreducible representation annihilated by a unipotent ideal.

In general, this definition is wrong: not all such representations are unitary. But sometimes it seems to work:

Almost Theorem

Every irreducible representation annihilated by a *special* unipotent ideal is unitary.

LMBM show that every special unipotent ideal is also a unipotent ideal.

Remark

Special unipotent representation = constituent of a unipotent Arthur packet

Unipotent ideals and unitarity

The Almost Theorem is true if, e.g.,

- $G_{\mathbb{R}}$ is complex classical (Barbasch),
- $G_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{H})$ (Vogan),
- $G_{\mathbb{R}}$ is real classical (Barbasch-Ma-Sun-Zhu),
- $G_{\mathbb{R}}$ is real exceptional (Adams-Miller-van Leeuwen-Vogan).

All representations annihilated by a unipotent ideal $I(\tilde{\mathbb{O}})$ are known to be unitary if:

- $G_{\mathbb{R}}$ is complex classical (Losev-Mason-Brown-Matvieievskyi),
- $G_{\mathbb{R}}$ is real exceptional and $\tilde{\mathbb{O}} = \mathbb{O}$ is a rigid orbit (Mason-Brown-Matvieievskyi).

Goal

Give general sufficient conditions for all representations annihilated by $I(\tilde{\mathbb{O}})$ to be unitary.

Conditions on orbits

Fix $\mathbb{O} \subset \mathfrak{g}^*$ be a nilpotent G-orbit and $\tilde{\mathbb{O}} \to \mathbb{O}$ a finite G-equivariant cover.
Let $\theta \colon G \to G$ be the Cartan involution (so $K_{\mathbb{R}} = G_{\mathbb{R}}^{\theta}$).

Definition

1. We say that $\tilde{\mathbb{O}}$ is *θ-stable* if $\theta(\mathbb{O}) = \mathbb{O}$ and $\theta^*\tilde{\mathbb{O}} \cong \tilde{\mathbb{O}}$.
2. We say that $\mathbb{O}$ has *small K-boundary* if

$$\mathrm{codim}_{\bar{\mathbb{O}} \cap (\mathfrak{g}/\mathfrak{k})^*}(\partial\mathbb{O} \cap (\mathfrak{g}/\mathfrak{k})^*) \geq 2,$$

   where $\mathfrak{k} = \mathrm{Lie}(K)$.

Note: if $G_{\mathbb{R}}$ is complex or $\mathrm{codim}_{\bar{\mathbb{O}}}\, \partial\mathbb{O} > 2$ then $\mathbb{O}$ automatically has small K-boundary.

Main theorem

Theorem (D.-Mason-Brown, in preparation)

Let $\tilde{\mathbb{O}} \to \mathbb{O} \subset \mathfrak{g}^*$ be a *Galois* cover of a nilpotent orbit and V an irreducible $(\mathfrak{g}, K)$-module annihilated by $I(\tilde{\mathbb{O}})$. Assume that $\tilde{\mathbb{O}}$ is θ-stable and $\mathbb{O}$ has small K-boundary. Then:

1. There exists a K-orbit $\mathbb{O}_K \subset \mathbb{O} \cap (\mathfrak{g}/\mathfrak{k})^*$ and a K-equivariant vector bundle $\mathcal{V}$ on $\mathbb{O}_K$ such that
$$V \cong \Gamma(\mathbb{O}_K, \mathcal{V}),$$
as K-representations.
2. V is unitary.

Remarks

- When $\tilde{\mathbb{O}} = \mathbb{O}$, θ-stability is automatic.
- We have a similar result when $\tilde{\mathbb{O}} = \mathbb{O}$ is birationally rigid.
- The Galois assumption is probably unnecessary.

Special case: complex groups

Take $G_{\mathbb{R}} = G'(\mathbb{C})$, so $G = G' \times G'$, $\theta =$ swap.
$(\mathfrak{g}, K)$-module = Harish-Chandra bimodule for G'.
Nilpotent orbit covers: $\tilde{\mathbb{O}} \times \tilde{\mathbb{O}}' \subset (\mathfrak{g}')^* \times (\mathfrak{g}')^*$.
Small K-boundary: automatic.
θ-stability: $\tilde{\mathbb{O}} = \tilde{\mathbb{O}}'$.

Corollary

Every Harish-Chandra bimodule annihilated on both sides by a unipotent ideal $I(\tilde{\mathbb{O}})$ is unitary. In particular, every special unipotent representation of a complex group is unitary.

Proof

By LMBM, we may assume without loss of generality that $\tilde{\mathbb{O}}$ is birationally rigid. In this case, $I(\tilde{\mathbb{O}}) = I(\mathbb{O}^{univ})$, where $\mathbb{O}^{univ}$ is the universal cover of $\mathbb{O}$. Since $\mathbb{O}^{univ} \to \mathbb{O}$ is Galois, the theorem applies.

Idea behind the proof

The unipotent ideals are defined using *filtered quantisation*. This theory has almost nothing to say about unitarity, but a lot to say about *good filtrations*.

There's an a priori unrelated theory linking good filtrations and unitarity:

"Theorem" (D.-Vilonen, conjectured by Schmid-Vilonen)

Every irreducible $(\mathfrak{g}, K)$-module V with real infinitesimal character has a canonical good filtration $F_\bullet V$, the *Hodge filtration*, such that

1. $F_\bullet V$ has excellent (homological) properties, and
2. $F_\bullet V$ detects whether V is unitary or not.

We proved the main theorem by connecting these two stories.

Recollection: Beilinson-Bernstein localisation

Let

- H be the abstract Cartan of G,
- $\mathfrak{h} = \mathrm{Lie}(H)$, and
- $\mathcal{B}$ be the flag variety of G.

Then to $\lambda \in \mathfrak{h}^*$, we can associate:

- An infinitesimal character

$$\chi_\lambda \colon Z(U(\mathfrak{g})) \cong S(\mathfrak{h})^W \to \mathbb{C},$$

- A sheaf of twisted differential operators $\mathcal{D}_\lambda$ on $\mathcal{B}$, with an isomorphism

$$\Gamma(\mathcal{B}, \mathcal{D}_\lambda) \cong U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} \mathbb{C}_{\chi_\lambda}.$$

Assume that λ is *integrally dominant*, i.e., $\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}_{<0}$ for all positive roots $\check{\alpha}$.

Theorem (Beilinson-Bernstein)

Let $\mathcal{M} \in \mathsf{Mod}(\mathcal{D}_\lambda)$. Then

1. We have
$$\mathrm{H}^i(\mathcal{B}, \mathcal{M}) = 0 \quad \text{for } i > 0.$$
2. If λ is also regular then the $\mathcal{D}_\lambda$-module $\mathcal{M}$ is globally generated.

Hence, for λ regular and integrally dominant, the functor

$$\Gamma \colon \mathsf{Mod}(\mathcal{D}_\lambda) \to \mathsf{Mod}(U(\mathfrak{g}))_{\chi_\lambda}$$

is an equivalence of categories.

Upshot: Every irreducible $(\mathfrak{g}, K)$-module with infinitesimal character χ_λ is of the form $\Gamma(\mathcal{M})$ for a unique irreducible $\mathcal{M} \in \mathsf{Mod}(\mathcal{D}_\lambda, K)$.

Enter Hodge theory

Mixed Hodge modules

Six functor formalism underlying classical Hodge theory:

X	$\mapsto$	$\mathsf{MHM}(X)$
variety over $\mathbb{C}$		category of $\mathcal{D}$-modules with Hodge structure

Key extra structure on a Hodge module $\mathcal{M}$:

- A good filtration $F_\bullet \mathcal{M}$, called the *Hodge filtration*,
- A finite filtration $W_\bullet \mathcal{M}$, called the *weight filtration*, with pure subquotients,
- When $\mathcal{M}$ is pure, a distribution-valued Hermitian form S, called a *polarisation* S.

Example: $X = \mathrm{pt}$

A *pure Hodge structure of weight w* is a finite dimensional complex vector space V with a decomposition

$$V = \bigoplus_{p+q=w} V^{p,q}.$$

The *Hodge filtration* is

$$F_\bullet V = \sum_{p \geq -\bullet} V^{p,w-p}.$$

A *polarisation* on V is a Hermitian form S such that the Hodge decomposition is orthogonal and S is $(-1)^q$-definite on $V^{p,q}$.

Example: $X = \mathbb{C}$, $\mathcal{M} = \mathbb{C}[z, z^{-1}]z^{\mu}$, $\mu \in \mathbb{R}$

This is a mixed Hodge module (pure if $\mu \notin \mathbb{Z}$) with Hodge filtration

$$F_p\mathcal{M} = \begin{cases} \mathrm{span}\{z^{\mu+n} \mid \mu + n \geq -p-1\}, & \text{if } p \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Localisation for Hodge modules

Now take $X = \mathcal{B}$ flag variety of G. If $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ is real, then we have a category $\mathrm{MHM}(\mathcal{D}_\lambda)$ of mixed Hodge $\mathcal{D}_\lambda$-modules.

Theorem (D.-Vilonen)

Assume $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ is real dominant and $\mathcal{M} \in \mathrm{MHM}(\mathcal{D}_\lambda)$. Then:

1. $\mathrm{H}^i(\mathcal{B}, F_p\mathcal{M}) = 0$ for $i > 0$ and all p,
2. If $\mathcal{M}$ is globally generated as a $\mathcal{D}_\lambda$-module, then so is $F_\bullet\mathcal{M}$.

Upshot for $(\mathfrak{g}, K)$-modules

Let V be an irreducible $(\mathfrak{g}, K)$-module with real infinitesimal character. Then V has a Hodge filtration

$$F_\bullet V = \Gamma(F_\bullet\mathcal{M}),$$

where $V = \Gamma(\mathcal{M})$ for $\mathcal{M} \in \mathrm{MHM}(\mathcal{D}_\lambda, K)$ irreducible, λ dominant. The theorem ensures good properties of $F_\bullet\mathcal{M}$ carry over to $F_\bullet V$.

Connection to unitarity

Let V be an irreducible $(\mathfrak{g}, K)$-module with real infinitesimal character.

Proposition (e.g., Adams-van Leeweun-Trapa-Vogan)

The $(\mathfrak{g}, K)$-module V is Hermitian if and only if it admits a Cartan involution $\theta\colon V \to V$, compatible with the Cartan involution $\theta\colon G_\mathbb{R} \to G_\mathbb{R}$ such that $K_\mathbb{R} = G_\mathbb{R}^\theta$.

Theorem (D.-Vilonen)

Assume V is Hermitian. Then V is unitary if and only if $\pm\theta$ acts on $\mathrm{Gr}_p^F V$ by $(-1)^p$ for all p.

Idea: Invariant form = Integral of polarisation twisted by θ.

The proof uses a version of the `atlas` algorithm to compute the signature of this integral.

How does this help with unipotent representations?

If V is a $(\mathfrak{g}, K)$-module and $F'_\bullet V$ is any good filtration then $\mathrm{Gr}^{F'} V$ is a K-equivariant $S(\mathfrak{g}/\mathfrak{k})$-module, i.e., a K-equivariant coherent sheaf on $(\mathfrak{g}/\mathfrak{k})^*$.

Lemma

Assume V is Hermitian and $\mathrm{Gr}^F V$ is indecomposable. Then V is unitary.

Proof

- Define $T\colon \mathrm{Gr}^F V \to \mathrm{Gr}^F V$ by $T(v) = (-1)^p \theta(v)$ for $v \in \mathrm{Gr}^F_p V$.
- Then T is an involution as a K-equivariant $S(\mathfrak{g}/\mathfrak{k})$-module.
- Hence $T = \pm 1$, so V is unitary.

Generic indecomposability

We want to find situations where we can apply the lemma. The first step is to look at the generic behaviour of $\mathrm{Gr}^F V$.

Definition

Let V be an irreducible $(\mathfrak{g}, K)$-module. Choose any good filtration $F'_\bullet V$. The *associated cycle of V* is

$$\mathrm{AC}(V) = \sum_{\mathbb{O}_K} [\mathrm{Gr}^{F'} V|_{\mathbb{O}_K}] \in \bigoplus_{\mathbb{O}_K} \mathrm{K}(\mathbb{O}_K) \cdot \mathbb{O}_K,$$

where the sum is over maximal K-orbits in $\mathrm{Supp}\,\mathrm{Gr}^F V$ and $\mathrm{K} =$ Grothendieck group of K-equivariant vector bundles.

$\mathrm{AC}(V)$ is independent of the choice of good filtration $F'_\bullet V$.

Theorem (D.-Mason-Brown)

Assume that $\mathbb{O}$ has small K-boundary, $\tilde{\mathbb{O}} \to \mathbb{O}$ is a θ-stable Galois cover and $I(\tilde{\mathbb{O}}) \cdot V = 0$. Then V is Hermitian and $\mathrm{AC}(V)$ is irreducible.

The Cohen-Macaulay property

In the setting of the main theorem, the previous theorem implies that any summand of $\mathrm{Gr}^F V$ must be supported on the boundary of $\mathbb{O}$. The hardest part of the proof is to show that there can be no such summands.

Theorem (D.-Mason-Brown)

Assume that V is annihilated by a unipotent ideal $I(\tilde{\mathbb{O}})$. Then $\mathrm{Gr}^F V$ is a Cohen-Macaulay $S(\mathfrak{g})$-module.

The Cohen-Macaulay property ensures that all summands have full-dimensional support.

More generally, the Cohen-Macaulay property holds for any weakly unipotent maximal ideal: a case by case check shows that all unipotent ideals have these properties.

Proof of the main theorem

Corollary

Assume that V is annihilated by a unipotent ideal and $\mathrm{AC}(V) = [\mathcal{V}] \cdot \mathbb{O}_K$ is irreducible. Then

1. The restriction map
$$\mathrm{Gr}^F V \to \Gamma(\mathbb{O}_K, \mathcal{V})$$
is injective, and an isomorphism if $\operatorname{codim} \partial\mathbb{O}_K \geq 2$.
2. $\mathrm{Gr}^F V$ is indecomposable.

In particular, if V is Hermitian then it is unitary.

Combining with the theorem on associated cycles, this proves the main theorem.

Remarks on the non-Galois case

Suppose $\tilde{\mathbb{O}} \to \mathbb{O}$ is not Galois (only occurs in exceptional types).

Problems

1. $\mathrm{AC}(V)$ need not be irreducible.
2. Not clear if V is Hermitian

Solution to 1

Define a *refined* associated cycle

$$\widehat{\mathrm{AC}}(V) \in \bigoplus_{\hat{\mathbb{O}}_K \subset \hat{\mathbb{O}}} \mathrm{K}(\hat{\mathbb{O}}_K) \cdot \hat{\mathbb{O}}_K, \quad \hat{\mathbb{O}} := \tilde{\mathbb{O}}/\mathrm{Gal}(\tilde{\mathbb{O}}/\mathbb{O}).$$

We can show that this is irreducible and hence (probably) that

$$\mathrm{Gr}^F V \cong \Gamma(\hat{\mathbb{O}}_K, \mathcal{V}).$$

Solution to 2?

This is less clear. It's equivalent to $\theta^* \widehat{\mathrm{AC}}(V) = \widehat{\mathrm{AC}}(V)$.

Proof of the Cohen-Macaulay property

Definition

An $S(\mathfrak{g})$-module M is *Cohen-Macaulay* if

$$\mathbb{D}M := \mathrm{RHom}_{S(\mathfrak{g})}(M, S(\mathfrak{g}))$$

is concentrated in a single homological degree.

Consider $M = \mathrm{Gr}^F V$, where V is annihilated by $I(\tilde{\mathbb{O}})$. By the vanishing theorem for the Hodge filtration,

$$\mathrm{Gr}^F V = \mathrm{R}\Gamma(\mathrm{Gr}^F \mathcal{M}),$$

for $\mathcal{M} \in \mathrm{MHM}(\mathcal{D}_\lambda, K)$ and λ dominant. Hence, by Grothendieck-Serre duality and a fundamental result of Saito,

$$\mathbb{D}\,\mathrm{Gr}^F V = \mathrm{R}\Gamma(\mathbb{D}\,\mathrm{Gr}^F \mathcal{M}) = \mathrm{R}\Gamma(\mathrm{Gr}^F \mathbb{D}\mathcal{M}),$$

up to a cohomological shift, where $\mathbb{D}\mathcal{M} \in \mathrm{MHM}(\mathcal{D}_{-\lambda}, K)$ is the Hodge module dual of $\mathcal{M}$.

Proof of the Cohen-Macaulay property

Example: $\tilde{\mathbb{O}} = \mathbb{O}_{\mathrm{prin}}$ the principal nilpotent orbit

$I(\mathbb{O}_{\mathrm{prin}})$ is the ideal with infinitesimal character zero. So $I(\mathbb{O}_{\mathrm{prin}}) \cdot V = 0$ implies $\lambda = 0$. So $\mathbb{D}\mathcal{M} \in \mathrm{MHM}(\mathcal{D}_0, K)$ and hence

$$\mathbb{D}\,\mathrm{Gr}^F V = \mathrm{R}\Gamma(\mathrm{Gr}^F \mathbb{D}\mathcal{M}),$$

is concentrated in one degree by the vanishing theorem.

In general, $-\lambda$ will not be dominant, so the vanishing theorem fails. To deal with the non-dominant twist, we apply *intertwining functors*

$$\mathrm{D}^b\mathrm{MHM}(\mathcal{D}_{-\lambda}) \to \mathrm{D}^b\mathrm{MHM}(\mathcal{D}_{-w_0\lambda}).$$

Weak unipotence of the annihilator implies that this commutes with $\mathrm{R}\Gamma(\mathrm{Gr}^F(-))$. Maximality implies that the intertwining functors don't introduce extra cohomology.

Thank you for listening. . . and
Happy Birthday Chengbo!