What does the unitary dual look like?

David Vogan

Massachusetts Institute of Technology

Real Reductive Groups and the Theta Correspondence

Yunnan July 22, 2024

Outline

Slides eventually at
`http://www-math.mit.edu/~dav/paper.html`

What's this about really?

$G(\mathbb{R})$ real reductive algebraic group.

$\widehat{G(\mathbb{R})}_u$ = (equiv classes of) irr unitary reps of $G(\mathbb{R})$.

I'll assume that studying this set (unitary dual) is the most world's best problem.

How can you approach it?

I'll start by answering the question in the title.

$G(\mathbb{R}) \rightsquigarrow$ {finite set of compact polyhedra U_j}.

Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

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This is two lines, two half lattices, and one interval.

Picture for $SL(2,\mathbb{R})$ found by Valentine Bargmann in 1947.

For those with OCD or PhD: more words are needed to make this precise. Example: nonsph princ ser at 0 is sum of two irreps: nonsph(pt, 0, 0) = hol ds(pt, 0, 0) + antihol ds(pt, 0, 0).

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So what do we need to do?

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$$\widehat{G(\mathbb{R})}_u = \coprod_j \; U_j \times V_j \times C_j.$$

Describe $\widehat{G(\mathbb{R})}_u \leftrightsquigarrow$ describe cpt polyhedra U_j.

Vec space V_j, cone-in-lattice C_j important but easy.

Main question today: what do cpt polyhed U_j look like?

Answer: U_j is finite union of product of simplices.

Goals for today:

1. say what kinds of simplices are allowed
2. recall work of Barbasch, (Barbasch and his friends) giving beautiful precise list of simplices in many cases
3. say how atlas software computes ugly precise list of simplices in all cases

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Remind me about the Weyl group...

G cplx conn red alg group $\supset B$ Borel $\supset H$ max torus.

(X^* alg chars of H) $\supset$ (R roots) $\supset$ (Π simple roots).

(X_* alg cochars) $\supset$ ($R^\vee$ coroots) $\supset$ ($\Pi^\vee$ simple coroots).

Based root datum of G is $(X^*, \Pi, X_*, \Pi^\vee)$, $\mathfrak{h}^*_{\mathbb{R}} = X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

$\mathfrak{h}^*_{\mathbb{R}}$ is the real vector space where the classical root system lives.

Root hyperplanes are $E_\alpha = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) = 0\}$ (each α in R).

Each root α defines simple reflection: $\mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$,

$$s_\alpha(\gamma) = \gamma - \gamma(\alpha^\vee) \cdot \alpha, \quad s_\alpha(\alpha) = -\alpha, \quad s_\alpha = \text{identity on } E_\alpha.$$

Weyl group of G is W = group generated by all s_α.

The open positive Weyl chamber is the open simplicial cone

$$C^+ = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} | \gamma(\alpha^\vee) > 0 \quad (\alpha \in \Pi \ \text{ simple})\}.$$

A Weyl chamber in $\mathfrak{h}^*_{\mathbb{R}}$ is a subset $w \cdot C^+$ (some $w \in W$).

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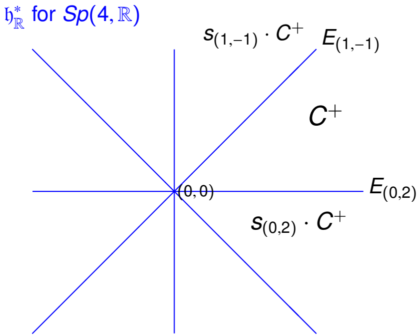

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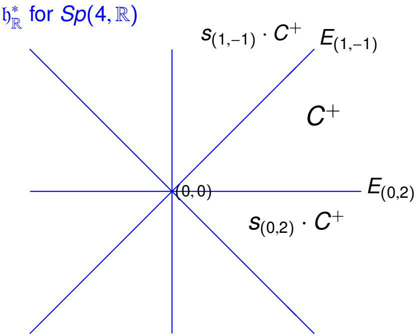

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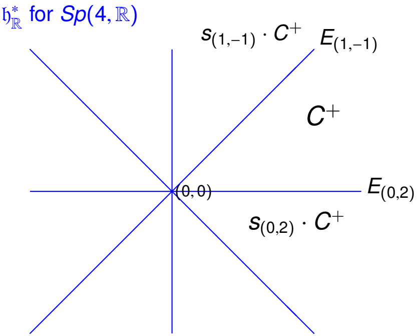

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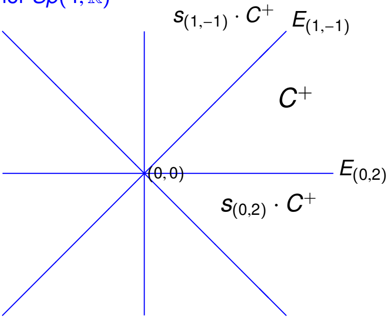

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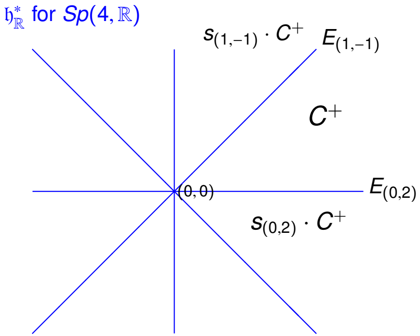

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$$\mathcal{A}^+ = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} | \gamma(\alpha^\vee) + m > 0 \quad ((\alpha^\vee, m) \in \Pi^{\vee,\text{aff}} \text{ simple})\}$$
$$= \{\gamma \in C^+ \mid \gamma(\alpha_0^\vee) < 1\}.$$

An alcove in $\mathfrak{h}^*_{\mathbb{R}}$ is a subset $w \cdot \mathcal{A}^+$ (some $w \in W^{\text{aff}}$).

And the affine Weyl group?

Based root datum of G is $(X^*, \Pi, X_*, \Pi^\vee)$, $\mathfrak{h}^*_{\mathbb{R}} = X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

Aff coroots are $R^{\vee,\text{aff}} = \{(\alpha^\vee, m) \mid \alpha^\vee \in R^\vee, m \in \mathbb{Z}\}$.

Pos aff coroots are $R^{\vee,\text{aff},+} = \{(\alpha^\vee, m) \mid m > 0 \text{ or } \alpha^\vee \in R^{\vee,+}, m = 0\}$.

Write $\alpha_0^\vee =$ lowest coroot (unique since G simple).

Simple aff coroots are $\Pi^{\vee,\text{aff}} = \{(\alpha^\vee, 0) \mid \alpha^\vee \in \Pi^\vee\} \cup \{(\alpha_0^\vee, 1)\}$.

Aff hyperplanes $E_{\alpha,m} = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) + m = 0\}$.

aff coroot $\rightsquigarrow$ simple aff reflection: $\mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$,

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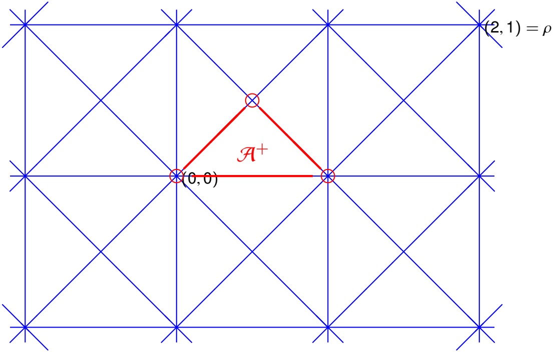
What do alcoves look like?

$\mathfrak{h}^*_{\mathbb{R}}$ for $Sp(4, \mathbb{R})$



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Action of W^{aff} on alcoves is simply transitive

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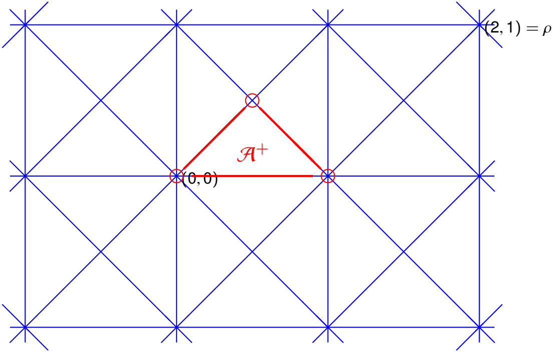

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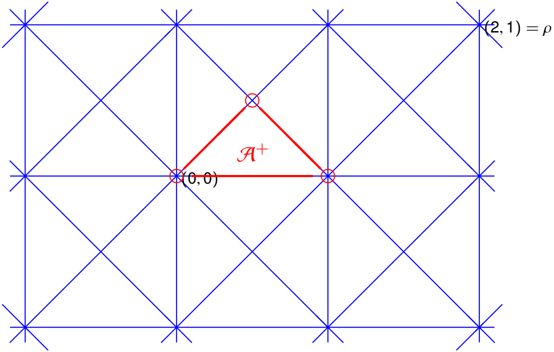

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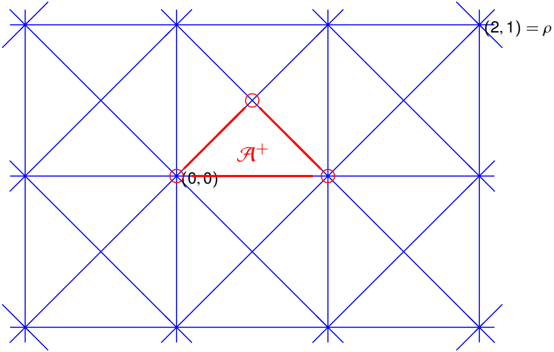

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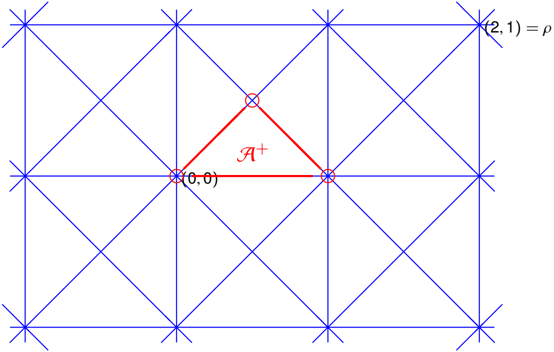

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What good are all these faces?

Langlands classif: irrs of real infl character indexed by

1. discrete parameter $(x, \lambda) \approx$ lowest K-type
2. continuous parameter $\gamma =$ infinitesimal character.

Here $x =$ KGB element: orbit of $K(\mathbb{C})$ on Borels in $G(\mathbb{C})$.

Finite # of x: 3 for $SL(2, R)$, 201 for $Sp(8, \mathbb{R})$, 320206 for split E_8.

Given x, set of allowed λ is finite # of cones in lattices

Given (x, λ), set of allowed γ is affine space $V_{\mathbb{R}}(x, \lambda) \subset \mathfrak{h}^*_{\mathbb{R}}$.

Therefore $V_{\mathbb{R}}(x, \lambda)$ is disjoint union of faces.

Theorem (Speh-V) Fix discrete parameter (x, λ).

1. If $\gamma_1, \gamma_2 \in$ same face of $V_{\mathbb{R}}(x, \lambda)$, then irr reps $J(x, \lambda, \gamma_1)$ and $J(x, \lambda, \gamma_2)$ are both unitary or both nonunitary.
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What does that say about the unitary dual?

Corollary Set $\widehat{G(\mathbb{R})}_{u,\text{real}}$ = unitary reps of real infl char. Then

$$\widehat{G(\mathbb{R})}_{u,\text{real}} = \bigcup_{x \in KGB} \bigcup_{\substack{\lambda \text{ allowed} \\ \text{for } x}} U(x, \lambda)$$

Claim in introduction:

$G(\mathbb{R}) \rightsquigarrow$ {finite set of compact polyhedra U_j}.

Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

Polyhedra $U(x, \lambda)$ are the U_j in the introduction.

Extending Cor to all infl chars gives real vector spaces V_j.

Given x, λ's are finite union of cones in lattices C_j.

To prove Claim, need to show $U(x, \lambda)$ is nearly independent of λ.

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$$\widehat{G(\mathbb{R})}_u = \coprod_j \; U_j \times V_j \times C_j.$$

Polyhedra $U(x,\lambda)$ are the U_j in the introduction.

Extending Cor to all infl chars gives real vector spaces V_j.

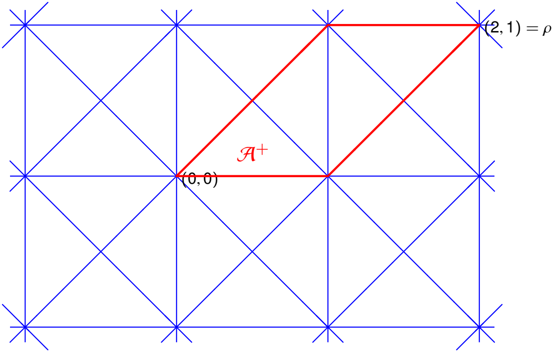
Given x, λ's are finite union of cones in lattices C_j.

To prove Claim, need to show $U(x,\lambda)$ is nearly independent of λ.

To describe unitary dual, need to compute all $U(x,\lambda)$.

What's the FPP...

FPP $\subset \mathfrak{h}^*_{\mathbb{R}}$ for $Sp(4, \mathbb{R})$



fundamental parallelepiped $= \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 \leq \gamma(\alpha^\vee) \leq 1 \mid (\alpha \in \Pi)\}$

Union of $\#W/\#Z(G_{sc})$ alcoves.

$G(\mathbb{R})$	# alcoves	# faces
$SL(2, \mathbb{R})$	1	3
$Sp(4, \mathbb{R})$	4	19
split E_8	696729600	2416970476

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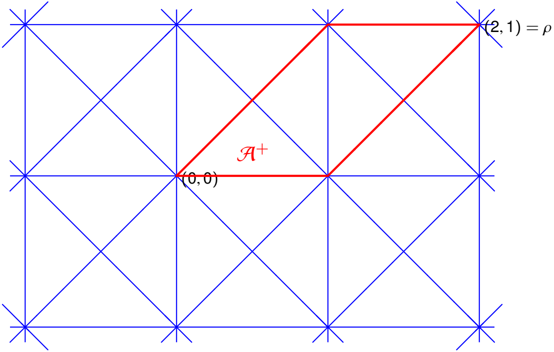

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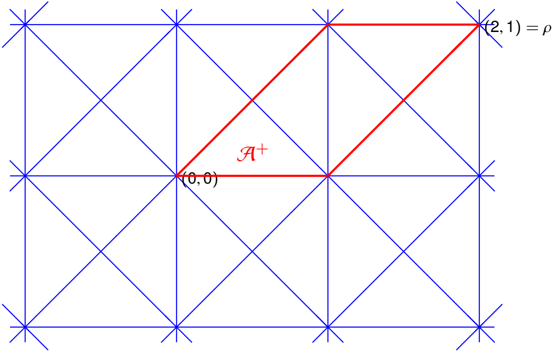

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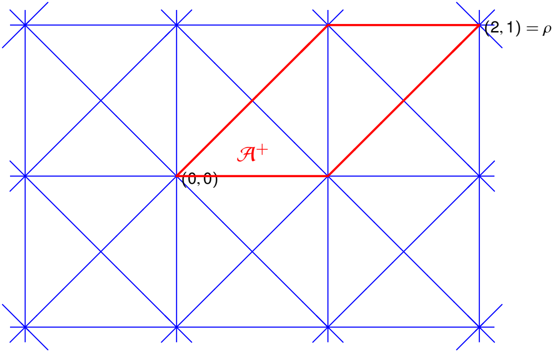

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green line is allowed infl chars γ.

unitary part is vertices $(2 + m_0, m_0)/2$, edges $\{(1 + t, t) \mid t \in (1 + m_1/2, 1 + (m_1 + 1)/2)\}$, some m_0 and m_1 in $\mathbb{N}$

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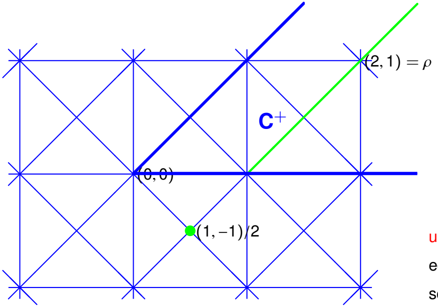

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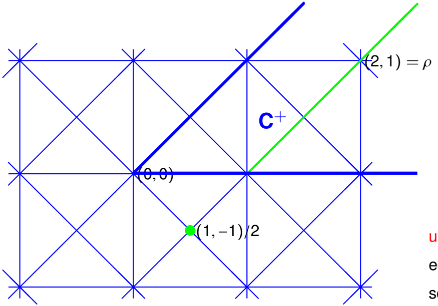

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Suppose (x, λ, γ) is a real Langlands parameter of infinitesimal character γ.

FPP conjecture is distilled from work of Dan Barbasch.

Define $S(\gamma) = \{\alpha \in \Pi \mid \gamma(\alpha^\vee) \leq 1\}$, a set of simple roots,

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Conclusion: assuming conjecture, unitary dual is known if we compute (finitely many) $U_{FPP}(x, \lambda)$, the FPP infl characters for unitary reps in the series (x, λ).

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