LOCAL THETA LIFTS OF ONE-DIMENSIONAL REPRESENTATIONS

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Abstract. Theta correspondence is a powerful tool to study representations of classical groups. Theta lifts of one-dimensional representations are extensively studied in the past. Although these representations are relatively simple in many senses, they could have deep applications. I would like to review some well known properties and also present some new results about these representations.

1. Introduction

In this note, I will focus on some representations of real classical groups constructed by local theta correspondence from one-dimensional representations. Although these representations are relatively simple, knowledge on these representations could lead deep results and could shade light on general situations.

Here, I will review some facts which are already well known to the experts but may be not completely obvious. I will discuss some recent results in joint works with Hung Yean Loke and U-Liang Tan and my thesis. These results exhibit the compatibility of theta lifting with taking associated cycles and derived functor constructions, which both play importent roles in the representations theory of real reductive groups.

Motivated by these results, I will ask some questions (which may not initially proposed by the author) on the theta lifts of genreal representations.

We will only consider the Type I reductive dual pairs, although most of the results also hold for Type II dual pairs. Real Lie groups are denoted by uppercase roman letters and the complexifications of their Lie algebras are denoted by the corresponding lowercase German letters. Subscript $_{\mathbb{C}}$ denote the complexification of a real Lie group or Lie algebra.

I will apologize for the possible incompleteness of references provided here due to the limitation of my knowledge.

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2. local Theta correspondence smooth and algebraic version

In this section, we review Howe's definition [How89b] of (local) theta correspondence (over $\mathbb{R}$). We will follow Howe's notation.

Let W be a symplectic space over $\mathbb{R}$, $\mathrm{Sp} := \mathrm{Sp}(W)$ be the symplectic group of W. Fix a unitary character of $\mathbb{R}$ and let ω be the corresponding oscillator (Segal-Shale-Weil) representation of the metaplectic group $\widetilde{\mathrm{Sp}}$, which could be realize on the Hilbert space of L^2 functions on $\mathbb{R}^{\dim_{\mathbb{R}} W/2}$. Let $\mathscr{Y}^\infty$ denote the space of smooth vectors, which could be identified with the space of Schwarze functions on $\mathbb{R}^{\dim_{\mathbb{R}} W/2}$. Fixing a maximal compact subgroup U of Sp is equivalent to fix a complex structure on W together with a compatible positive definite Hermitian form on this complex vector space. We view W as a complex vector space $W^{\mathbb{C}}$ of dimension $\dim_{\mathbb{R}} W/2$. Now U is the unitary group preserving the Hermitian form. Then $\mathscr{Y}$, the space of U-finite vectors, is isomorphic to $\mathcal{P}(W^{\mathbb{C}})$, the space of polynomials on $W^{\mathbb{C}}$.

Let (G, G') be a *real reductive dual pair* in the symplectic group Sp. So G, G' act on W reductively and are mutual centralizers. One always can choose a maximal compact subgroup U of Sp such that $K = \mathrm{U} \cap G$ and $K' = \mathrm{U} \cap G'$ are maximal compact subgroups of G and G' respectively. We fix this choice of U and so maximal compact subgroups K and K' from now on.

For a subgroup G of Sp, $\widetilde{G}$ denote its preimage in $\widetilde{\mathrm{Sp}}$. We always assume that $G \cap \mathrm{U}$ is a maximal compact subgroup of G. Define

$$\mathscr{R}(\widetilde{G}; \mathscr{Y}^\infty) = \left\{ \begin{array}{l} \text{continuous irreducible admissible } \widetilde{G}\text{-modules} \\ \text{which can be realized as a quotient of } \mathscr{Y}^\infty \end{array} \right\} \Big/ \begin{array}{l} \text{infinitesimal} \\ \text{equivalence} \end{array}$$

$$\mathscr{R}(\mathfrak{g}, \widetilde{K}; \mathscr{Y}) = \left\{ \begin{array}{l} \text{irreducible admissible } (\mathfrak{g}, K)\text{-modules which} \\ \text{can be realized as a quotient of } \mathscr{Y} \end{array} \right\} \Big/ \begin{array}{l} \text{infinitesimal} \\ \text{equivalence} \end{array}$$

Here $\mathscr{R}(\widetilde{G}; \mathscr{Y}^\infty)$ could be view as a subset of $\mathscr{R}(\mathfrak{g}, \widetilde{K}; \mathscr{Y})$.

For $\rho^\infty \in \mathscr{R}(\widetilde{G}; \mathscr{Y}^\infty)$ and $\rho \in \mathscr{R}(\mathfrak{g}; \widetilde{K}; \mathscr{Y})$, define

$$\Omega^\infty_{\mathscr{Y}^\infty, \rho^\infty} = \mathscr{Y}^\infty \Big/ \bigcap_{T \in \mathrm{Hom}_{\widetilde{G}}(\mathscr{Y}^\infty, \rho^\infty)} \mathrm{Ker}\,(T)\ ;$$

$$\Omega_{\mathscr{Y}, \rho} = \mathscr{Y} \Big/ \bigcap_{T \in \mathrm{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathscr{Y}, \rho)} \mathrm{Ker}\,(T)\ .$$

Now $\Omega_{\mathscr{Y}, \rho} = \Theta(\rho) \otimes \rho$ as $(\mathfrak{g}', \widetilde{K}') \times (\mathfrak{g}, \widetilde{K})$-module (see [MVW87, Chapter 2.III] or [Ma12b]). In [How89b] Howe shows that $\Theta(\rho)$ is a finite length $(\mathfrak{g}', K')$-module with infinitesimal character. Moreover, it has a unique irreducible quotient $\theta(\rho)$. We call $\Theta(\rho)$ the maximal Howe quotient of ρ.

Howe duality correspondence/local theta correspondence over $\mathbb{R}$:
$\rho \mapsto \theta(\rho)$ defines an one-one correspondence between $\mathscr{R}(\mathfrak{g}, \widetilde{K}; \mathscr{Y})$ and $\mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$.

Let ρ^∞ be a smooth Fréchet representation in sense of Casselman-Wallach and ρ be its Harish-Chandra module. As a consequence, $\Omega^\infty_{\mathscr{Y}^\infty, \rho^\infty} = \Omega^\infty(\rho^\infty) \hat{\otimes} \rho^\infty$, where $\Omega^\infty(\rho^\infty)$ is a smooth Fréchet representation and $\hat{\otimes}$ means projective tensor product.

The Harish-Chandra module of $\Theta^\infty(\rho^\infty)$ is a non-zero quotient of $\Theta(\rho)$ and $\Theta^\infty(\rho^\infty)$ has a unique irreducible quotient $\theta^\infty(\rho^\infty)$ whose Harish-Chandra module is $\theta(\rho)$. However, the relationship between $\Theta^\infty(\rho^\infty)$ and $\Theta(\rho)$ is not clear at least to the author.

Question 2.1. *Whether or when $\Theta(\rho)$ is the Harish-Chandra module of $\Theta^\infty(\rho^\infty)$? Moreover, when $\Theta(\rho)$(or $\Theta^\infty(\rho^\infty)$) is irreducible? How $\Theta(\rho)$(or $\Theta^\infty(\rho^\infty)$) decomposes into irreducible sub-quotients if it is reducible?*

When (G, G') is a compact dual pair, the theta correspondence is simple and could be deduced from classical invariant theory [KV78] [How89a]. Suppose G is compact, we have following decomposition

$$\mathscr{Y} = \bigoplus_{\sigma \in \mathscr{R}(\mathfrak{g}, \widetilde{G}; \mathscr{Y})} \sigma \otimes L(\sigma).$$

where $L(\sigma) = \Theta(\sigma) = \theta(\sigma)$ is an irreducible unitary lowest weight $(\mathfrak{g}', \widetilde{K}')$-module with lowest $\widetilde{K}'$-type σ'. This decomposition is a basic tool for the study of Howe correspondence.

Another important observation is following diamond dual pairs for (irreducible) non-compact dual pair (G, G'), see Figure 1. Here K (resp. K', $M^{(1,1)}$, $M'^{(1,1)}$) is the maximal compact subgroup of G (resp. G', M, M') and the pairs of groups similarly placed in the two diamonds are reductive dual pairs. G (resp. G') is a symmetric subgroup of M (resp. M'). We always let $\mathfrak{p}$ (resp. $\mathfrak{p}'$) be the non-compact part in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since M is Hermitian symmetric, the non-compact part of $\mathfrak{m}$ (resp. $\mathfrak{m}'$) is decomposed into two $M^{(1,1)}$ (resp. $M'^{(1,1)}$) invariant spaces, say $\mathfrak{m}^{(2,0)}$ and $\mathfrak{m}^{(0,2)}$ (resp. $\mathfrak{m}'^{(2,0)}$ and $\mathfrak{m}'^{(2,0)}$). Fact 3 in [How89b] says, as subspaces of $\mathfrak{sp}$,

$$\mathfrak{m}^{(2,0)} \oplus \mathfrak{p} = \mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)}. \tag{1}$$

Similar equation holds for $\mathfrak{p}'$.

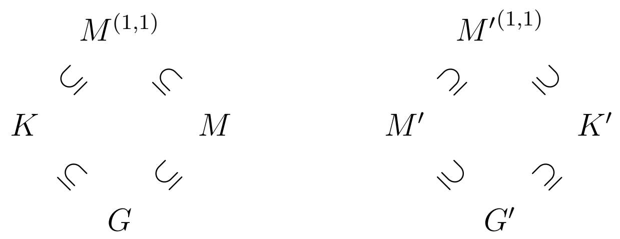

FIGURE 1. Diamond dual pairs

3. THE K-SPECTRUM OF THE MAXIMAL HOWE QUOTIENT

From now on we switch the role of G and G'. Let $\rho' \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$ be a genuine one dimensional $\widetilde{G}'$-module. In this case, the Harish-Chandra module of $\Theta^\infty(\rho^\infty)$ is exactly $\Theta(\rho)$, following from an automatic continuity theorem due to van den Ban-Delorme and Brylinski-Delorme [BD92]. I learnt a generalized version from [Sun11, Theorem 3.1]. So, we focus on the algebraic version of theta correspondence.

Now we summarize some results on the K-spectrum of $\Theta(\rho')$.

Proposition 3.1. *Let (G, G') be a type I reductive dual pair. For any character $\rho' \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$, the maximal Howe quotient $\Theta(\rho)$ is $\widetilde{K}$-multiplicity free.*

$$\Theta(\rho') = \bigoplus_{\sigma \in \mathscr{R}(\mathfrak{k}, \widetilde{K}; \mathscr{Y})} m_\sigma \sigma$$

where the multiplicity $m_\sigma \leq 1$ and $m_\sigma = 1$ only if

$$\operatorname{Hom}_{\widetilde{K}'}(\sigma', \rho') \neq 0 \tag{2}$$

(σ' is the lowest $M'^{(1,1)}$-type of $L(\sigma) = \theta(\sigma)$.

In following two cases, σ occur in $\Theta(\rho')$ if and only if (2) is satisfied.

(i) ρ' is an unitary character and (G, G') is in the stable range with G' smaller. In this case, $\Theta(\rho') = \theta(\rho')$ is irreducible and unitarizable (c.f. [ZH97] *or see* [Ma12b]*).*

(ii) G is split and ρ' corresponde to the trivial representation of G'. In this case, $\Theta(\rho')$ could be embedded into certain degenerate principle series representation of G.

The structure of degenerate principle series and maximal Howe quotients of trivial representations have been well study more or less simultaneously by many authors (c.f. [KR90] [Zhu92] [HL99] [Lee96] [LZ97] [LZ98] [LZ08] [LZ08] [Yam11]). One may read a recent note [LZ12] on a complete description for $(G, G') = (\widetilde{\mathrm{Sp}}(2n, \mathbb{R}), \mathrm{O}(p, q))$. The explicit structure results could have deep applications, for a recent application on the first occurrence conjecture of Kudla-Rallis, see [SZ12].

4. Geometry of theta lifts

In this section, we will discuss theta lifts in a geometric point of view.

4.1. **Natural Filtration.** Since $\mathscr{Y} = \mathcal{P}(W^{\mathbb{C}})$ is an polynomial ring, it has a nature notion of degree. The degree of a $\widetilde{K}$-type is defined to be the lowest degree j such that it occur in $\mathcal{P}^j(W^{\mathbb{C}})$ by Howe. Fix a lowest degree $\widetilde{K}'$-type σ' of ρ' with degree j_0. Let $\eta\colon \mathscr{Y} \to \Theta(\rho') \otimes \rho'$ be the natural quotient map. Denote the σ'-isotypic component of a $\widetilde{K}'$-module V by $V[\sigma']$. Then the σ'-isotypic component of $\eta(\mathscr{Y}_{j_0})$ satisfies

$$\eta(\mathscr{Y}_{j_0})[\sigma'] = V_\sigma \otimes V_{\sigma'},$$

where $V_\sigma \subset \Theta(\rho')$ (resp. $V_{\sigma'} \subset \rho'$) is irreducible $\widetilde{K}$(resp. $\widetilde{K}'$)-module of type σ(resp. σ'). Moreover, σ is a lowest degree $\widetilde{K}$-type of $\Theta(\rho')$ and it generats $\Theta(\rho')$ under $\mathfrak{g}$-action.

In the maximal quotient $\Theta(\rho')$, lowest degree $\widetilde{K}$-type(s) is multiplicity free (see [He00, Theorem 13 (5)]). In some cases, lowest $\widetilde{K}$-type is agree with minimal $\widetilde{K}$-type in sense of Vogan. Here are some questions about lowest degree $\widetilde{K}$-types.

Question 4.2. *When lowest $\widetilde{K}$-type and minimal $\widetilde{K}$-type agree? Note that minimal $\widetilde{K}$-type of an irreducible $(\mathfrak{g}, \widetilde{K})$-module may not be unique. Is lowest degree $\widetilde{K}$-type unique?*

Now we continue to define a filtration on $\Theta(\rho')$. Fix a minimal degree $\widetilde{K}$-type σ, $F_j = \mathcal{U}_j(\mathfrak{g})V_\sigma$ defines an exhaustive filtration on $\Theta(\rho')$. This filtration is a good filtration (c.f. [Vog91]), i.e. the graded module

$$\operatorname{Gr}\Theta(\rho') = \bigoplus_j F_j/F_{j-1} \tag{3}$$

is a finite generated $\widetilde{K}$-equivariant $\mathcal{S}(\mathfrak{p})$-module. On the other hand, the filtration is natural, since it is same as the filtration on $\Theta(\rho')$ inherited from the natural filtration on $\mathscr{Y}$, i.e. we have (c.f. [NZ04, Section 3.3] or [LMT11a])

$$\eta(\mathscr{Y}_{2j+j_0})[\sigma'] = F_j \otimes V_{\sigma'}.$$

4.3. **Moment maps.** To interpret the formula (proposition 3.1) on the K-spectrum of the theta lift of unitary character in stable range geometrically, we first recall the explicit definition of moment maps. For the compact dual pair (M, K'), we have a natural homomorphism of $\mathbb{C}$-algebras from classical invariant theory:

$$\phi^*\colon \mathcal{S}(\mathfrak{m}^{(2,0)}) \to \mathbb{C}[W^{\mathbb{C}}]^{K'_{\mathbb{C}}},$$

where $\mathcal{S}(\mathfrak{m}^{(2,0)})$ denote the symmetric algebra of $\mathfrak{m}^{(2,0)}$ and $\mathbb{C}[W^{\mathbb{C}}]^{K'_{\mathbb{C}}}$ denote the subring of $K'_{\mathbb{C}}$-invariants in $\mathbb{C}[W^{\mathbb{C}}]$. On the other hand, the projection of $\mathfrak{p}$ onto $\mathfrak{m}^{(2,0)}$ component via (1) is an isomorphism. We identify $\mathfrak{p}$ and $\mathfrak{m}^{(2,0)}$ via this isomorphism, and get

$$\phi^*\colon \mathcal{S}(\mathfrak{p}) \to \mathbb{C}[W^{\mathbb{C}}]^{K'_{\mathbb{C}}}. \tag{4}$$

ϕ^* determine a map $\phi\colon W^{\mathbb{C}} \to \mathfrak{p}^*$ between affine schemes. Similarly, we have $\phi'\colon W^{\mathbb{C}} \to \mathfrak{p}'^*$. Keep following diagram in mind.

$$\mathfrak{p}^* \xleftarrow{\ \phi\ } W^{\mathbb{C}} \xrightarrow{\ \phi'\ } \mathfrak{p}'^*$$

We always identify $W^{\mathbb{C}}$, $\mathfrak{p}$ and $\mathfrak{p}'^*$ with certain spaces of matrices. We would like to point out that ϕ is well studied in the context of classical invariant theory. Let $W^{\mathbb{C}}/K'_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[W^{\mathbb{C}}]^{K'_{\mathbb{C}}}$ be the affine quotient of $W^{\mathbb{C}}$ by $K'_{\mathbb{C}}$. One can rephrase basic results of classical invariant theory [Wey46][GW09] geometrically as following:

1. ϕ factor through the affine quotient $W^{\mathbb{C}} \to W^{\mathbb{C}}/K'_{\mathbb{C}}$ by (4);

2. Fist Fundamental Theorem: $W^{\mathbb{C}}/K'_{\mathbb{C}}$ is a (reduced) closed subscheme of $\mathfrak{p}^*$, i.e.
$$\phi(W^{\mathbb{C}}) = W^{\mathbb{C}}/K'_{\mathbb{C}} \subset \mathfrak{p}^*;$$
3. Second Fundamental Theorem: $W^{\mathbb{C}}/K'_{\mathbb{C}}$ is either the whole $\mathfrak{p}^*$ or a determinantal variety which is the intersection of zero locuses of determinates of all minors of certain size. Moreover, the radical ideal of the determinantal variety is generated by the determinates of all minors of certain size.

Now we focus on the case that (G, G') is in the stable range with G' smaller. In this case, ϕ' has many good properties, for example ϕ' is flat and its worst fiber — the null fiber $\mathcal{N} := \phi'^{-1}(0)$ is reduced and normal. Moreover, $\mathcal{N}$ has a dense $K_{\mathbb{C}}$-orbit. Nishiyama observed these phenomena in [NOZ06] [Nis07] and show that the notion of theta lifting of nilpotent orbits could be defined formally possessing some good geometric properties.

Recall one of the equivalent definitions of nilpotent orbit (see for example, [PV94]). Suppose a reductive affine algebraic group E act linearly on a vector space V. An E-orbit is called nilpotent if its closure contains 0. Let $\mathfrak{N}_E(V)$ be the set of nilpotent orbits.

For a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}' \in \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$, $\phi(\phi^{-1}(\overline{\mathcal{O}'}))$ contains an unique open $K_{\mathbb{C}}$-orbit $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$. Let $\theta(\mathcal{O}') = \mathcal{O}$ and call it the theta lift of $\mathcal{O}'$. The map
$$\theta \colon \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*) \to \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*),$$
is injective. We also denote θ to be the linear extension of above map to spaces of formal combinations of nilpotent orbits.

This map has been studied by many authors, for example [Oht91],[DKP05] and also (originally) appare as resolution of singularities [KP82]. If we parametrize the nilpotent $K_{\mathbb{C}}$ orbit in $\mathfrak{p}^*$ by signed Young diagram, θ correspond to a column adding operation.

4.4. **Graded $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module of $\Theta(\rho')$.** Twisting by certain fixed genuine character ς(resp. ς') of $\widetilde{K}$(resp. $\widetilde{K}'$), we can reduce $\widetilde{K}$-action into linear actions of K, i.e. as $K \times K'$-module,
$$\varsigma \otimes \mathscr{Y} \otimes \varsigma' = \mathbb{C}[W^{\mathbb{C}}]$$
where $K \times K'$ act on $W^{\mathbb{C}}$ linearly and act on $\mathbb{C}[W^{\mathbb{C}}]$ by translation. A priori, the graded module $\operatorname{Gr}\Theta(\rho')$ is an $(\mathcal{S}(\mathfrak{p}), \widetilde{K})$-module. Under above twisting, we can view $\operatorname{Gr}\Theta(\rho')$ as a $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module with $\mathcal{S}(\mathfrak{p})$-module structure unchanged.

Let $\rho' \in \mathscr{R}(\widetilde{G}'; \mathscr{Y}^\infty)$ be an unitary genuine character of $\widetilde{G}'$. So we are in case (i) of proposition 3.1. Mainly by comparing the K-spectrum, one can show that
$$\operatorname{Gr}\theta(\rho') = \left(\mathbb{C}[W^{\mathbb{C}}] \otimes_{\mathcal{S}(\mathfrak{p}')} \chi\right)^{K'_{\mathbb{C}}} = \left(\mathbb{C}[\mathcal{N}] \otimes_{\mathbb{C}} \chi\right)^{K'_{\mathbb{C}}}. \tag{5}$$
Here $\mathcal{S}(\mathfrak{p})$ act on $\mathbb{C}[W]$ and the null-cone $\mathcal{N} := \phi'^{-1}(0)$ via ϕ^*; $\chi = \rho'^* \otimes \varsigma'$ is a $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-character with trivial $\mathcal{S}(\mathfrak{p}')$-action.

4.5. **Invariants.** First recall the definition of some invariants [Vog91]. For simplicity, let V be a finite length $(\mathfrak{g}, K)$-module. Fixing a good filtration $\{ F_j \}$ of V, let $\mathscr{V}$ be the corresponding associated coherent sheaf on $\mathfrak{p}^*$ of the graded $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module $\operatorname{Gr} V = \bigoplus_j F_j/F_{j-1}$. Denote the support of $\mathscr{V}$ by $\mathcal{V}(V) \subset \mathfrak{p}^*$. $\mathcal{V}(V)$ is called the *associated variety* of V, which is an invariant independent of the choice of good filtration. Since $\operatorname{Gr} V$ is $K_{\mathbb{C}}$-equivariant and V has generalized infinitesimal characters, $\mathcal{V}(V)$ is a union of unipotent $K_{\mathbb{C}}$-orbits. For each open $K_{\mathbb{C}}$-orbit contained in $\mathcal{V}(V)$, define their multiplicities to be the dimension of the fiber of $\mathscr{V}$ at a point in these orbits (may need taking some graded module of $\mathscr{V}$ first, or equivalently look at the length of $\mathscr{V}$ localizing on a generic point in the $\mathcal{V}(V)$). Define the *associated cycle* of V to be the formal sum of the closure of these open orbits with their multiplicity and denote it by $\operatorname{AC}(V)$.

Now we retain the natural filtration defined in section 4.1. Localize the finitely generated $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module $\operatorname{Gr}\theta(\rho')$ and let $\mathscr{A}$ be the associated coherent sheaf on $\mathfrak{p}^*$. One can reinterpret (5) as following:

$$\mathscr{A} = (\phi_*\phi'^*\mathscr{A}')^{K'_{\mathbb{C}}} = ((\phi|_{\mathcal{N}})_*\mathscr{O}_{\mathcal{N}} \otimes \chi)^{K'_{\mathbb{C}}}. \tag{6}$$

Here $\mathscr{O}_{\mathcal{N}}$ denote the structure sheaf on $\mathcal{N}$, χ denote the constant sheaf on $\mathcal{N}$. When $\mathscr{M}$ is a sheaf of $K'_{\mathbb{C}}$-modules on $\mathfrak{p}^*$, $(\mathscr{M})^{K'_{\mathbb{C}}}$ means the sheaf obtained by taking $K'_{\mathbb{C}}$-invariant of $\mathscr{M}(U)$ for each open set $U \subset \mathfrak{p}^*$.

By some algebraic geometric yoga, the geometric properties of moment maps and (6) will lead following theorem, which is more or less known to exports, for example see [Prz93], [NZ04], [PT07] and [Yan11].

Theorem 4.6 ([LMT11a] [Ma12b]). *Let (G, G') be a real reductive dual pair in stable range with G' smaller. Let ρ' be a genuine unitary character of $\widetilde{G}'$. Let $\mathcal{O} := \theta(\mathcal{O}')$, where $\mathcal{O}' := \{ 0 \}$ is the zero orbit in $\mathfrak{p}'^*$. Then*

(i) $\mathcal{V}(\theta(\rho')) = \overline{\mathcal{O}}$;
(ii) $\mathrm{AC}(\theta(\rho')) = [\overline{\mathcal{O}}] = \theta(\mathrm{AC}(\rho'))$, *i.e. the multiplicity of $\mathcal{O}$ is one;*
(iii) As $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module,

$$\operatorname{Gr}\theta(\rho') = \operatorname{Ind}_{K_x}^{K_{\mathbb{C}}}\chi_x.$$

Here $x \in \mathcal{O}$, $K_x = \operatorname{Stab}_{K_{\mathbb{C}}}(x)$ whose Levi part contains a factor isomorphic to $K'_{\mathbb{C}}$, $\chi_x = \chi \circ \alpha$ and $\alpha\colon K_x \to K'_{\mathbb{C}}$ is the "projection" morphism.

Although the notion of unipotent representation is not properly defined, unipotent representations are expected to satisfies many properties. By Theorem 4.6, one can check that $\theta(\rho')$ satisfies the expected relationships between its infinitesimal character and its associative variety. For a proper choice of ρ', usually the trivial representation, (iii) means that $\theta(\rho')$ satisfies the K-spectrum equation expected by Vogan [Vog91, Conjecture 12.1] for unipotent representation and it is the "quantization" of the nilpotent orbit $\mathcal{O} \subset \mathfrak{p}^*$, see also [Yan11].

5. Transfers of K-types and local theta lifting

We consider following two families of dual pairs listed in Table 1 and Table 2.

	G	$G'^{p,q}$	H	K	ρ	$\rho_{p,q}$
Case $\mathbb{R}$	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{O}(p,q)$	$\mathrm{U}(r,s)$	$\mathrm{O}(m)$	$\det^\epsilon$	$1^{\xi,\eta}$
Case $\mathbb{C}$	$\mathrm{U}(n,n)$	$\mathrm{U}(p,q)$	$\mathrm{U}(r,s)\times\mathrm{U}(s,r)$	$\mathrm{U}(m)$	1	1
Case $\mathbb{H}$	$\mathrm{O}^*(2n)$	$\mathrm{Sp}(p,q)$	$\mathrm{U}(r,s)$	$\mathrm{Sp}(m)$	1	1

Here $\xi \equiv \epsilon - (s-q)$, $\eta \equiv \epsilon - (r-p) \pmod 2$; $\det^\epsilon$ is the trivial or determinate representation of $\mathrm{O}(m)$ depends on the parity of ϵ; $\mathbf{1}^{\xi,\eta}$ is the characters of $\mathrm{O}(p,q)$ whose restriction on $\mathrm{O}(p)\times\mathrm{O}(q)$ is $\det^\xi \otimes \det^\eta$. Note ρ and $\rho_{p,q}$ are all unitary characters of $G'^{p,q}$, whose restriction on $\mathfrak{g}'$ is trivial. $(G, G'^{p,q})$ is in stable range if and only if $n \geq p+q$ in case $\mathbb{R}$ and case $\mathbb{C}$, $n \geq 2(p+q)$ in case $\mathbb{H}$.

Table 1. Family A of dual pairs

	$G^{p,q}$	G'	Stable range	j_0
Case $\mathbb{R}$	$\mathrm{O}(p,q)$	$\mathrm{Sp}(2n,\mathbb{R})$	$p,q \geq 2n$,$\max\{ p,q \} > 2n$	nr
Case $\mathbb{C}$	$\mathrm{U}(p,q)$	$\mathrm{U}(n_1,n_2)$	$p,q \geq n_1+n_2$	$(n_1+n_2)r$
Case $\mathbb{H}$	$\mathrm{Sp}(p,q)$	$\mathrm{O}^*(2n)$	$p,q \geq n$	$2nr$

Table 2. Family B of dual pairs

Consider the theta lifts of trivial representations from $G'^{p,q}$ to G for pairs in Table 1. Let $\theta^{p,q}(1)$ be the theta lifts. It is well known that $\{ \theta^{p,q}(1) \}$ are different representations of $\widetilde{G}$ [Li89]. But they share some interesting properties:
(i) they have same the infinitesimal character;
(ii) they have the same complex associated variety which is the closure of a $G_{\mathbb{C}}$-orbit $\mathcal{O}_{\mathbb{C}}$ in $\mathfrak{g}^*$ and their associated varieties (via Kostant-Sekiguchi correspondence) give all different real forms of $\mathcal{O}_{\mathbb{C}}$.
(iii) all of them are unitarizable;
(iv) possible embeddings of $\theta^{p,q}$ into certain degenerate principle series representation gives all irreducible submodules of this degenerate principle series representation. (see [LZ12, Theorem 6.1]).

Now let us discuss yet another type of relations between these representations exhibiting certain compatibility between theta lifting and derived functor constructions.

5.1. **Transfers of K-types.** Let $G_{\mathbb{C}}$ be a complex reductive Lie group with complex Lie algebra $\mathfrak{g}$. Let G_1 and G_2 be two real forms of $G_{\mathbb{C}}$. Let K_i be maximal compact subgroup in G_i. We always assume the Cartan involutions and complex conjugations with respect to $\mathrm{Lie}(G_i)$ commute with each other. Let $\mathscr{C}(\mathfrak{g}, K_i)$ denote the category of $(\mathfrak{g}, K)$-modules. Composing the forgetful functor

$$\mathcal{F}^{\mathfrak{g},K_1\cap K_2}_{\mathfrak{g},K_1} \colon \mathscr{C}(\mathfrak{g}, K_1) \to \mathscr{C}(\mathfrak{g}, K_1 \cap K_2)$$

and the derived functors of Zuckerman functor $\Gamma^{\mathfrak{g},K_2}_{\mathfrak{g},K_1\cap K_2}$

$$\mathrm{R}^j\Gamma^{\mathfrak{g},K_2}_{\mathfrak{g},K_1\cap K_2} \colon \mathscr{C}(\mathfrak{g}, K_1 \cap K_2) \to \mathscr{C}(\mathfrak{g}, K_2),$$

gives a family of functors

$$\Gamma^j := \mathrm{R}^j\Gamma^{\mathfrak{g},K_2}_{\mathfrak{g},K_1\cap K_2} \circ \mathcal{F}^{\mathfrak{g},K_1\cap K_2}_{\mathfrak{g},K_1} \colon \mathscr{C}(\mathfrak{g}, K_1) \to \mathscr{C}(\mathfrak{g}, K_2).$$

Applying Γ^j on $(\mathfrak{g}, K_1)$-modules could construct $(\mathfrak{g}, K_2)$-modules. This procedure is called *transfer of K-types*. It is have been studied in [EPWW85] [Fra91] [Wal94] and [WZ04].

5.2. **Relationships with theta lifting.** The relationships between transfers of K-types and local theta lifting was speculated in [Fra91] and a very precise conjecture was made in [WZ04, Conjecture 5.1], which is the case $\mathbb{R}$ in Theorem 5.4. Following theorem provided an positive answer to their speculation.

Theorem 5.3 ([Ma12a] [Ma12b])**.** *We adapt the notation in Table 1. Let G_1, G_2 be two subgroups of $G_{\mathbb{C}}$ both are isomorphic to G such that the analytic subgroup with Lie algebra $\mathrm{Lie}(K_2)_{\mathbb{C}} \cap \mathrm{Lie}(G_1)$ in G_1 is isomorphic to H. Let $(G_1, G'^{m,0})$ be in the stable range, i.e. $n \geq m$.*
Then

$$\Gamma^j\theta^{m,0}(\rho) = \bigoplus \Gamma_{p,q}\theta^{m,0}(\rho) \quad \text{and} \quad \Gamma_{p,q}\theta^{m,0}(\rho) \cong \theta^{p,q}(\rho_{p,q}),$$

Here $\Gamma_{p,q}\theta^{m,0}(\rho)$ is certain submodule of $\Gamma^j\theta^{m,0}(\rho)$. p, q run over the subset of non-negative integers such that: i) $p + q = m$; ii) $p \leq r, q \leq s$ in Case $\mathbb{R}$ and Case $\mathbb{C}$; $2p \leq r, 2q \leq s$ in Case $\mathbb{H}$; iii) $j = rs - (r - p)(s - q)$ in Case $\mathbb{R}$; $j = 2(rs - (r - p)(s - q))$ in Case $\mathbb{C}$; $j = rs - (r - 2p)(s - 2q)$ in Case $\mathbb{H}$.

Since $\widetilde{G} \cong \widetilde{G}_1 \cong \widetilde{G}_2$, the above theorem could be view as a construction of $\widetilde{G}$-modules from unitary lowest weight $\widetilde{G}$-modules. If we consider the derived functor Γ^j in all degrees together, there is a pretty formula:

$$\bigoplus_{j\in\mathbb{N}} \Gamma^j\theta^{m,0}(\rho) \cong \bigoplus_{p+q=m} \theta^{p,q}(\rho_{p,q}).$$

Furthermore, we have another family of examples, which transfers representations between different real reductive groups.

Theorem 5.4 ([LMT11b] [Ma12a] [Ma12b]). *We adapt the notation in Table 2. Assume $p+q$ is even in case $\mathbb{R}$ and case $\mathbb{C}$. Let $G_1 \cong G^{p,q}$, $G_2 \cong G^{p+r,q-r}$ such that the analytic subgroup with Lie algebra $\mathrm{Lie}(K_2)_{\mathbb{C}} \cap \mathrm{Lie}(G_1)$ in G_1 isomorphic to $G^{p,r} \times G^{0,q-r}$. Let $\theta^{p,q}(1)$ be the theta lift of the trivial representation of G'.*

(i) If $(G^{p+r,q-r}, G')$ is outside the stable range, then $\Gamma^* \theta^{p,q}(1) = 0$.
(ii) If $(G^{p+r,q-r}, G')$ is in the stable range, then

$$\Gamma^j \theta^{p,q}(1) = \begin{cases} \theta^{p+r,q-r}(1) & \text{if } j = j_0 \\ 0 & j < j_0, \end{cases}$$

where $j_0 = nr, (n_1+n_2)r$ and $2nr$ in case $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ respectively as listed in Table 2. If $j > j_0$, $\Gamma^j \theta^{p,q}(1)$ is a direct sum of several copies of $\theta^{p+r,q-r}(1)$.

While these phenomenons has not been complete understood yet. At least, one can ask following questions.

Question 5.5. *1. How to understand these representations in the context of local Langlands, for example in terms of Langland/Arthur/Vogan packets?*
2. Is there any geometric reason behind?

We briefly discuss the key ingredients for the proof of above theorems. Fist, we calculate the $\widetilde{K}_2$-spectrums of the derived functor modules. Then we compair the $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2}$ actions on the $\widetilde{K}_2$ isotopic component. To achieve this, we observe an identity on the images of Hecke-algebras (which is also observed in [LNW08] for different purpose)

$$\omega(\mathcal{U}(\mathfrak{g})^{K_{\mathbb{C}}}) = \omega(\mathcal{U}(\mathfrak{m}')^{G'_{\mathbb{C}}}).$$

This identity is independent of real forms and help us converting the calculation of $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2}$ actions on $\widetilde{K}_2$-isotypic components to $\mathcal{U}(\mathfrak{m}')^{G'_{\mathbb{C}}}$ actions on ρ'. The latter one is manageable by a theorem of Helgason on invariant differential operators (similar to the case of $(\mathfrak{g}, K)$-modules with scaler K-type, see [Zhu03]).

6. Theta lifts beyond one-dimensional representations

We retain notations in Table 2. Let μ be a genuine $\widetilde{G}^{s,0}$-module, which is finite dimensional since $G^{s,0}$ is compact. Let $L(\mu) = \theta(\mu)$ be the theta lift of μ from $\widetilde{G}^{s,0}$ to $\widetilde{G}'$, which is an unitary lowest weight module. In this section, we discuss $\theta^{p,q}(L(\mu))$ the theta lift of $L(\mu)$ from $\widetilde{G}'$ to $\widetilde{G}^{p,q}$. $\theta^{p,q}(L(\mu))$ is beyond theta lifts of one-dimensional representations. The below theorems is proved by reducing to theta lifts of character via the observation (see. [LL06])

$$\theta^{p,q}(L(\mu)) = \left(\theta^{p,q+s}(1) \otimes \mu\right)^{\widetilde{G}^{s,0}}.$$

We always assume that $(G^{p,q+s}, G')$ is in the stable range.

Theorem 6.1 ([NZ04] [LMT11a] [Ma12b]). *Let $G = G^{p,q}$, $\theta = \theta^{p,q}$. $q \geq n$ in Case $\mathbb{R}$, $q \geq n_1, n_2$ in Case $\mathbb{C}$ and $2q \geq n$ in Case $\mathbb{H}$. Let $\mathcal{O} = \theta(\mathcal{O}')$, where $\overline{\mathcal{O}'}$ is the associated variety of $L(\mu)$. Then*

(i) $\mathcal{V}(\theta(L(\mu)) = \overline{\mathcal{O}}$;
(ii) $\mathrm{AC}(\theta(L(\mu)) = \theta(\mathrm{AC}(L(\mu))$, i.e. the multiplicity is preserved by theta lifting;

(iii) *as* $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$*-module*

$$\mathrm{Gr}\,\theta^{p,q}(L(\mu)) \cong \mathrm{Ind}_{K_x}^{K_{\mathbb{C}}}\chi_x,$$

if $n > s$ *in case* $\mathbb{R}$, $n_1, n_2 > s$ *in case* $\mathbb{C}$, $n > 2s$ *in case* $\mathbb{H}$ *(so* $L(\mu)$ *is singular).*

Here $x \in \mathcal{O}$, $K_x = \mathrm{Stab}_{K_{\mathbb{C}}}(x)$, χ_x *is a twisting of* $\mu \circ \beta$ *and* β *is certain surjective map from* K_x *to* $G_{\mathbb{C}}^{s,0}$.

Theorem 6.2 ([LMT11b] [Ma12b]). *Let* Γ^j *be the derived functor transfers* $\widetilde{G}^{p,q}$*-module to* $\widetilde{G}^{p+r,q-r}$ *module as in Theorem 5.4. Then, for* $0 < r \leq q$,

(i) *if* $(G^{p+r,q+s-r}, G')$ *is not in the stable range,* $\Gamma^*\theta^{p,q}(L(\mu)) = 0$;

(ii) *if* $(G^{p+r,q+s-r}, G')$ *is in the stable range,*

$$\Gamma^j\theta^{p,q}(L(\mu)) = \begin{cases} \theta^{p+r,q-r}(L(\mu)) & \text{if } j = j_0 \\ 0 & \text{otherwise,} \end{cases}$$

where $j_0 = nr, (n_1+n_2)r$ *and* $2nr$ *in case* $\mathbb{R}, \mathbb{C}$ *and* $\mathbb{H}$ *respectively. If* $j > j_0$, $\Gamma^j\theta^{p,q}(1)$ *is a direct sum of several copies of* $\theta^{p+r,q-r}(1)$.

Theorem 6.1 and Theorem 6.2 motivate us to ask the last question in this note: how to generize our results?

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