TWO TOPICS ON LOCAL THETA CORRESPONDENCE

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety.

I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

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Contents

SUMMARY

This thesis contains two topics on local theta correspondence.

The first topic is on the relationship between derived functor modules and local theta correspondences. Derived functor construction can transfer representations between different real forms of a complex Lie group. On the other hand, representations of different real forms also could be constructed by theta correspondences of different real reductive dual pairs (with same complexification). We first observe an equation on the image of Hecke-algebras for see-saw pair, $\omega(U(\mathfrak{g})^H) = \omega(U(\mathfrak{h}')^{G'})$, which generalize the correspondence of infinitesimal characters. Then, we use it to study the $U(\mathfrak{g})^K$-actions on the isotypic components of theta lifts and show that the derived (Zuckerman) functor modules of theta lifts of one dimensional representations (characters) are determined by their K-spectrums. We identify families of derived functor modules constructed in Enright(1985), Frajria(1991), Wallach(1994) and Wallach-Zhu (2004) with theta lifts of unitary characters. One can rephrase the results in following form: the derived functor modules of theta lifts of unitary characters are again (possibly direct sum of) theta lifts of (other) characters (of possibly another real form). By a restriction method, we also extend the theorem to theta lifts of unitary highest weight modules as in a joint work with Loke and Tang. All these results suggest that theta liftings and derived functors are compatible operations.

In the second topic, we study invariants of theta lifts. Fixing a good K-invariant filtration on a finite length $(\mathfrak{g}, K)$-module, the associated sheaf of corresponding graded module is a $K_{\mathbb{C}}$-equivariant coherent sheaf supported

on a union of nilpotent $K_{\mathbb{C}}$-orbit(s) in $\mathfrak{p}^*$. The fiber of the associated sheaf at a point in general position is a rational representation of its stabilizer in $K_{\mathbb{C}}$, called the isotropic representation at this point. The (genuine) virtual character of the isotropic representation is an invariant. We calculated the isotropic representations for theta lifts of unitary characters and unitary highest weight modules under certain natural filtrations. As corollaries, we recovered associated varieties and associated cycles of these representations. Our result show that, outside the stable range, sometimes theta lifting and taking associated cycle are compatible, while sometimes they are not compatible.

Furthermore, we show that some families of unitary representations, obtained by two step theta liftings, are "height-3" representations satisfying a prediction of Vogan: the K-spectrums are isomorphic to the spaces of global sections of certain $K_{\mathbb{C}}$-equivariant algebraic vector bundles defined by their isotropic representations.

Since our calculations also suggest that there could be a notion of "lifting" of isotropic representations compatible with theta lifting of representations. We propose a precise conjecture in the general cases, of an inductive nature. A positive answer to these questions may contribute to a better understanding of unipotent representations constructed by iterated theta liftings.

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Chapter 1

Introduction

In this thesis, we focus on the "singular" part of the set of irreducible representations of real classical groups. We study two topics both aim to understand the role of irreducible (unitary) representations constructed by local theta correspondence in the general theories of the representations of real reductive groups.

The first topic is on the relationship between certain derived functor constructions and local theta lifts. We studied the transfer of representations between different real forms of a complex classical Lie group via derived functors of Zuckerman functors. The main result is that the derived functor module of the theta lift (or, more generally, the irreducible component of the maximal Howe quotient) of a character is characterized by its K-spectrum (and its infinitesimal character).

The second topic is about the invariants of theta lifts. This part is build on a joint work with Loke and Tang [LMT11a]. We computed the isotropic representations of the theta lifts of unitary characters and unitary lowest weight modules under a natural good filtration. Then we recovered the Associated cycles of these representations. Furthermore, we showed that stable range double theta lifts of unitary characters are height-3 representations satisfying a prediction of Vogan: their K-spectrums are isomorphic to the spaces of global sections of certain $K_{\mathbb{C}}$-equivariant vector bundles defined by their isotropic representations.

In Chapter 2, we introduce notations and some necessary facts for later exploration. Most material in Chapter 2 may be known to experts. So the reader may safely skip this chapter at first and read it when we refer it in other chapters. In Chapter 3 and Chapter 4, we discuss above two topics respectively. For the statement and discussions of main results of each topics, see Introductions of these chapters.

Chapter 2

Preliminaries

2.1 Notation

We will introduce notation for the whole thesis, basically following Harish-Chandra's convention.

We use capital letters, for example G, denote real Lie groups. $\mathfrak{g}_0 = \mathrm{Lie}(G)$ denote the (real) Lie algebra of G and $\mathfrak{g} := (\mathfrak{g}_0)_{\mathbb{C}} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}_0$. K_G (or simply K) denote certain maximal compact subgroup of G. For real Lie group we always assume G is reductive. We follow Wallach's definition [Wal88, Section 2.1] of real reductive group. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ respect to K_G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexification of this decomposition. The universal algebra (over $\mathbb{C}$) of $\mathfrak{g}$ is denote by $\mathcal{U}(\mathfrak{g})$. The adjoint representations of G (resp. its derivative) on $\mathfrak{g}_0$, $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ are denoted by Ad (resp. ad). For real reductive Lie group G, $\widehat{G}$ denote the isomorphism class of irreducible admissible representations. For an isomorphism class σ of representation, V_σ denote a vector space realize σ; σ^* and V_σ^* denote their dual (contragredient). Sometimes we may simply write σ for V_σ, without explicitly fixing a realization of σ.

For a vector space V, the symmetric algebra of V is denote by $\mathcal{S}(V)$. If V is finite dimensional, $\mathbb{C}[V] \cong \mathcal{S}(V^*)$ denote the polynomial ring (ring of regular functions) on V. There has natural grading on $\mathcal{S}(V)$. $\mathcal{S}^d(V)$ denote the space of all elements with degree d and $\mathcal{S}_d(V)$ denote the space of all elements with degree $\leq d$.

Here *variety* means *abstract variety*, i.e. integral separated scheme of finite type over algebraically closed field k[1] (c.f. [Har77, Section II.4]). Since we will only study variety, we not distinguish algebraic subsets of

[1] We only use $\mathbb{C}$ actually.

variety and the corresponding reduced subschemes. The structure sheaf of a scheme X is denoted by $\mathscr{O}_X$, the stalk at $x \in X$ of a sheaf $\mathscr{L}$ is denoted by $\mathscr{L}_x$. In particular, the local ring at x is denoted by $\mathscr{O}_{X,x}$ (or simply $\mathscr{O}_x$). For an open set $U \subset X$, $\mathscr{L}(U)$ denote the space of sections on U. For any morphism $f\colon Y \to X$, f_* and f^* denote the direct image and inverse image functors. For a locally closed set $Z \subset X$, $i_Z\colon Z \to X$ denote the inclusion and $k[Z] = i_Z^*\mathscr{O}_X(Z)$ denote the ring of regular functions on Z.

For a variety X with G-action, we say G act linearly (or geometrically) on $k[X]$ if it act by the translation action induced from the G-action on X.

We will use boldface letter to denote an array of numbers. We will ignore zeros in the tail of an array of integers and write $(a_1, \cdots, a_k, 0, \cdots 0)$ by $(a_1, \cdots, a_k)$. Two array of numbers can be add or subtract coordinate-wise. $(\mathbf{a}, \mathbf{b})$ denote the array obtained by appending $\mathbf{b}$ to $\mathbf{a}$. $\mathbf{a}^r$ denote the array of integers by reverse the order of $\mathbf{a}$. An array of "1"(resp. "0") with length p is denoted by $\mathbf{1}_p$(resp. $\mathbf{0}_p$). We assign lexicographical order on the set of arrays and $\mathfrak{a} \geq 0$ means all entries of $\mathfrak{a}$ are non-negative.

$I_{n,m}$ denote the matrix of size $n \times m$ with 1 on the diagonal. $I_m := I_{m,m}$ denote the identity matrix of size $m \times m$.

2.2 $(\mathfrak{g}, K)$-module

Let $\mathfrak{g}$ be a complex Lie algebra and K be a compact Lie group such that $\mathfrak{k} = \mathrm{Lie}(K)_{\mathbb{C}}$ is a complex Lie subalgebra of $\mathfrak{g}$. The pair $(\mathfrak{g}, K)$ is a special case of *Harish-Chandra pairs*.

Definition 1. A $(\mathfrak{g}, K)$-*module* is a pair (π, V) with V a complex vector space, $\pi\colon \mathfrak{g} \cup K \to \mathrm{End}_{\mathbb{C}}(V)$ a representation of $\mathfrak{g}$ and K satisfying following conditions:

(1) $\dim \mathrm{span}\{\, \pi(\mathrm{K})\mathrm{v} \,\} < \infty$ for any $v \in V$;

(2) $\pi(k)\pi(X) = \pi(\mathrm{Ad}_k X)\pi(k)$ for all $k \in K, X \in \mathfrak{g}$;

(3) The action of K on V is continuos. The differential of K-action is the restriction of $\mathfrak{g}$-representation on $\mathfrak{k}$, i.e.

$$\pi(X)v = \lim_{t \to 0} \frac{1}{t}(\pi(\exp(tX))v - v) \quad \forall v \in V, X \in \mathfrak{k}_0.$$

Let $\mathscr{C}(\mathfrak{g}, K)$ be the category of $(\mathfrak{g}, K)$-module.

For any $\sigma \in \widehat{K}$, let $V(\sigma)$ be the σ-isotypic component of V. A $(\mathfrak{g}, K)$-module is an *admissible representation* if $V(\sigma)$ is finite dimension for all $\sigma \in \widehat{K}$.

For a continuous (under certain topology) representation $(\pi, \mathcal{V})$ of G , let

$$\mathcal{V}_K := \{\, v \in \mathcal{V} \mid \dim \mathrm{span} \{\, \pi(\mathrm{K})\mathrm{v} \,\} < \infty \,\}$$

be the space of K-finite vectors of $\mathcal{V}$, which is a $(\mathfrak{g}, K)$-module called the *Harish-Chandra module* of $\mathcal{V}$. By Harish-Chandra's theory, the Harish-Chandra module of irreducible unitary representation is an irreducible admissible $(\mathfrak{g}, K)$-module. Two irreducible unitary representations are isomorphic if and only if there Harish-Chandra modules are isomorphic. Moreover, every irreducible admissible $(\mathfrak{g}, K)$-module is the Harish-Chandra module of an irreducible Hilbert space representation. Since $(\mathfrak{g}, K)$-module play an importent role in the representation theory of real reductive groups, we will focus on $(\mathfrak{g}, K)$-modules.

Later we will use following theorems from Harish-Chandra, Lepowsky and McCollum [LM73].

Theorem 2 (c.f. [Wal88, Section 3.5.4 and Section 3.9])**.** *Let G be a real reductive group, K be its maximal compact subgroup.*

1. Let W be an admissible $(\mathfrak{g}, K)$-module, $\gamma \in \widehat{K}$. X be an $\mathcal{U}(\mathfrak{g})^K$ and K-invariant subspace of the γ isotypic component $W(\gamma)$. Then $(\mathcal{U}(\mathfrak{g})X)(\gamma) = X \subset W(\gamma)$.

2. Let V and W be two irreducible $(\mathfrak{g}, K)$-modules. Let $\gamma \in \widehat{K}$ such that $V(\gamma)$ and $W(\gamma)$ both nonzero. Then V and W are equivalent as $(\mathfrak{g}, K)$-module if and only if $V(\gamma)$ and $W(\gamma)$ are equivalent as $\mathcal{U}(\mathfrak{g})^K$-module.

2.3 Local Theta correspondence

In this section, we review Howe's definition [How89b] of (local) theta correspondence (over $\mathbb{R}$). We follow Howe's notation.

2.3.1 Reductive dual pairs

Let k be a local field, W be an symplectic space over k, $\mathrm{Sp}(W)$ be the symplectic group of W which is the subgroup of $\mathrm{GL}(W)$ preserves a non-degenerate symplectic form on W. A pair of subgroup (G, G') in $\mathrm{Sp}(W)$ is called *reductive dual pair* [How79b] over k, if

(i) G is centralizer of G' in $\mathrm{Sp}(W)$ and vice versa;

(ii) G and G' act on W absolute reductively, i.e. under any field extension, W decompose into direct sum of irreducible G-modules (or G'-modules).

Every reductive dual pair (G, G') can be decompose into direct sum of irreducible reductive dual pairs: it is a decomposition of symplectic space $W = \bigoplus_{i=1}^{s} W_i$ such $G = G_1 \times \cdots \times G_s$, $G = G'_1 \times \cdots \times G'_s$ and (G_i, G'_i) are irreducible reductive dual pairs in $\mathrm{Sp}(W_i)$. We listed irreducible reductive dual pairs over $\mathbb{C}$ (resp. $\mathbb{R}$) in Table 2.1 (resp. Table 2.2, where $\mathbb{H}$ is the set of quaternions).

	G	G'	$\mathrm{Sp}(W)$
Type I	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{O}(m, \mathbb{C})$	$\mathrm{Sp}(2nm, \mathbb{C})$
Type II	$\mathrm{GL}(n, \mathbb{C})$	$\mathrm{GL}(m, \mathbb{C})$	$\mathrm{Sp}(2nm, \mathbb{C})$

Table 2.1: Irreducible reductive dual pairs over $\mathbb{C}$

	G	G'	$\mathrm{Sp}(W)$
Type I	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{O}(p, q, \mathbb{R})$	$\mathrm{Sp}(2n(p+q), \mathbb{R})$
	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{O}(m, \mathbb{C})$	$\mathrm{Sp}(4nm, \mathbb{R})$
	$\mathrm{U}(r, s)$	$\mathrm{U}(p, q)$	$\mathrm{Sp}(2(p+q)(r+s), \mathbb{R})$
	$\mathrm{O}^*(2n)$	$\mathrm{Sp}(p, q)$	$\mathrm{Sp}(4n(p+q), \mathbb{R})$
Type II	$\mathrm{GL}(n, \mathbb{R})$	$\mathrm{GL}(m, \mathbb{R})$	$\mathrm{Sp}(2nm, \mathbb{R})$
	$\mathrm{GL}(n, \mathbb{C})$	$\mathrm{GL}(m, \mathbb{C})$	$\mathrm{Sp}(4nm, \mathbb{R})$
	$\mathrm{GL}(n, \mathbb{H})$	$\mathrm{GL}(m, \mathbb{H})$	$\mathrm{Sp}(8nm, \mathbb{R})$

Table 2.2: Irreducible reductive dual pairs over $\mathbb{R}$

From the classification of irreducible reductive dual pairs, or else, we have following observations. For any real symplectic space W, define $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ and extend the real symplectic form $\mathbb{C}$-linearly to $W_{\mathbb{C}}$. For real reductive dual pair (G, G') in $\mathrm{Sp}(W)$, let $G_{\mathbb{C}}$ and $G'_{\mathbb{C}}$ the complexification of G and G'. Then $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ form a complex dual pair in $\mathrm{Sp}(W_{\mathbb{C}})$. One the other hand, we call a real symplectic subspace W of $W_{\mathbb{C}}$ a *real from* of $W_{\mathbb{C}}$ if $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} W_{\mathbb{C}}$ and the symplectic form restricted on W is non-degenerate. Suppose $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ is a complex dual pair in complex symplectic group $\mathrm{Sp}(W_{\mathbb{C}})$, let $G = G_{\mathbb{C}} \cap \mathrm{Sp}(W)$ and $G' = G'_{\mathbb{C}} \cap \mathrm{Sp}(W)$. By a proper choice of real form W, (G, G') will be a real reductive dual pair in $\mathrm{Sp}(W)$. We call (G, G') a *real form* of $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ since $G, G', \mathrm{Sp}(W)$ are real froms of complex Lie group $G_{\mathbb{C}}, G'_{\mathbb{C}}, \mathrm{Sp}(W_{\mathbb{C}})$ respectively.

2.3.2 Definition of theta correspondence

Write Sp for the big symplectic group $\mathrm{Sp}(W)$ containing G and G'. $\widetilde{\mathrm{Sp}}$ denote the metaplectic cover of Sp. Fix a unitary character of $\mathbb{R}$, let ω be the oscillator representation of $\widetilde{\mathrm{Sp}}$ and $\mathscr{Y}^\infty$ be the space of smooth vectors. Denote $\mathscr{R}(\widetilde{E})$ the infinitesimal equivalente classes of continuous irreducible admissible representation of $\widetilde{E}$ on locally convex topological vector spaces. Let $\mathscr{R}(\widetilde{E};\mathscr{Y}^\infty)$ be the subset of $\mathscr{R}(\widetilde{E})$ which can be realized as a quotient of $\mathscr{Y}^\infty$ by an $\widetilde{E}$-invariant closed subspace.

For a reductive dual pair (G, G') in Sp, choose a maximal compact subgroup U of Sp such that $K = \mathrm{U}\cap G$ and $K' = \mathrm{U}\cap G'$ are maximal compact subgroups of G and G' respectively. Let $\mathscr{Y}$ be the space of $\widetilde{\mathrm{U}}$-finite vectors in $\mathscr{Y}^\infty$. For any subgroup E of G such that $K_E := E\cap \mathrm{U}$ is a maximal compact subgroup of E, let $\mathscr{R}(\mathfrak{e}, \widetilde{K}_E;\mathscr{Y})$ be the infinitesimal equivalent classes of irreducible $(\mathfrak{e}, \widetilde{K}_E)$-modules which can be realized as a quotient of $\mathscr{Y}$. All elements in $\mathscr{R}(\widetilde{E};\mathscr{Y}^\infty)$ and $\mathscr{R}(\mathfrak{e}, \widetilde{K}_E;\mathscr{Y})$ are genuine representations of the double covering in the sense that the centers of $\widetilde{E}$ and $\widetilde{K}$ act non-trivially.

Clearly taking Harish-Chandra module gives a inclusion $\mathscr{R}(\widetilde{E};\mathscr{Y}^\infty)\hookrightarrow \mathscr{R}(\mathfrak{e}, \widetilde{K}_E;\mathscr{Y})$. For $\rho\in\mathscr{R}(\widetilde{G};\mathscr{Y}^\infty)$ (view as smooth representation of $\widetilde{G}$ in the sense of Casselman-Wallach), let ρ_0 be the corresponding $(\mathfrak{g}, \widetilde{K})$-module. Define

$$\Omega^\infty_{\mathscr{Y}^\infty,\rho} = \mathscr{Y}^\infty \Big/ \bigcap_{T\in \mathrm{Hom}_{\widetilde{G}}(\mathscr{Y}^\infty,\rho)} \mathrm{Ker}\,(T)$$

and

$$\Omega_{\mathscr{Y},\rho_0} = \mathscr{Y} \Big/ \bigcap_{T\in \mathrm{Hom}_{\mathfrak{g},\widetilde{K}}(\mathscr{Y},\rho_0)} \mathrm{Ker}\,(T)\,.$$

By Lemma 5 in Section 2.3.3, as $(\mathfrak{g}, \widetilde{K})\times(\mathfrak{g}', \widetilde{K}')$-module,

$$\Omega_{\mathscr{Y},\rho_0}\cong \rho_0\otimes\Theta(\rho_0).$$

Howe [How89b] proved that $\Theta(\rho_0)$ is a finite length $(\mathfrak{g}', \widetilde{K}')$-module with infinitesimal character and it has a unique irreducible quotient $\theta(\rho_0)$. Note that the restriction to $\mathscr{Y}$ induces an injection

$$\mathrm{Hom}_{\widetilde{G}}(\mathscr{Y}^\infty,\rho)\to \mathrm{Hom}_{\mathfrak{g},\widetilde{K}}(\mathscr{Y},\rho_0).$$

Therefore, the space of $\widetilde{K}\times\widetilde{K}'$-finite vectors in $\Omega^\infty_{\mathscr{Y}^\infty,\rho}$ is a quotient of $\Omega_{\mathscr{Y},\rho_0}$

and

$$\Omega^{\infty}_{\mathscr{Y}^{\infty},\rho} = \rho \,\hat{\otimes}\, \Theta^{\infty}(\rho)$$

where $\Theta^{\infty}(\rho)$ is a finite length smooth $\widetilde{G}'$-module and $\hat{\otimes}$ denote projective tensor product[2]. Clearly, the Harish-Chandra module of $\Theta^{\infty}(\rho)$ is a non-zero quotient of $\Theta(\rho_0)$ and $\Theta^{\infty}(\rho)$ has a unique irreducible quotient $\theta^{\infty}(\rho)$ with Harish-Chandra module $\theta(\rho_0)$. However, the relationship between $\Omega^{\infty}_{\mathscr{Y}^{\infty},\rho}$ and $\Omega_{\mathscr{Y},\rho_0}$ are subtle. It is not known in general at least to the author.

Definition 3. We define the *theta lifting* map

$$\theta\colon \mathscr{R}(\mathfrak{g}, \widetilde{K}; \mathscr{Y}) \to \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$$

by $\rho_0 \mapsto \theta(\rho_0)$. We also have the smooth version of theta lifting map:

$$\theta^{\infty}\colon \mathscr{R}(\widetilde{G}; \mathscr{Y}^{\infty}) \to \mathscr{R}(\widetilde{G}'; \mathscr{Y}^{\infty})$$

defined by $\rho \mapsto \theta^{\infty}(\rho)$.

For any $\rho_0 \in \mathscr{R}(\mathfrak{g}, \widetilde{K}; \mathscr{Y})$, $\Theta(\rho_0)$ is usually called the *maximal Howe quotient*. $\rho_0 \mapsto \Theta(\rho_0)$ defines a map

$$\Theta\colon \mathscr{R}(\mathfrak{g}, \widetilde{K}; \mathscr{Y}) \to \mathscr{C}(\mathfrak{g}', \widetilde{K}')$$

whose image is in the subcategory of finite length $(\mathfrak{g}', \widetilde{K}')$-modules. We call Θ the *full theta lifting map*. Similarly, $\rho \mapsto \Theta^{\infty}(\rho)$ defines map

$$\Theta^{\infty}\colon \mathscr{R}(\widetilde{G}; \mathscr{Y}^{\infty}) \to \mathscr{C}(\widetilde{G}').$$

Here $\mathscr{C}(\widetilde{G}')$ denote the category of Casselman-Wallach $\widetilde{G}'$-representation and the image of Θ^{∞} is in the subcategory of finite length Casselman-Wallach $\widetilde{G}'$-representations..

Since the role of G and G' are symmetric, we will abuse notation by using same symbols for maps from $\widetilde{G}'$-modules to $\widetilde{G}$-modules. In this thesis, we will focus on the algebraic version of theta lifting, i.e. θ and Θ.

[2]Actually, both ρ and $\Theta(\rho)$ will be nuclear spaces, there is only one reasonable topological tensor product.

2.3.3 A lemma from Moeglin Vigneras and Waldspurger

In this section, we prove a lemma essentially[3][4] from Moeglin, Vigneras and Waldspurger [MVW87] for completeness. This lemma explains why $\Omega_{\mathscr{Y},\rho_0}$ is a tensor product.

At first, we review some basic properties of $(\mathfrak{g}, K)$-modules. For any $(\mathfrak{g}, K)$-module V, $V = \bigoplus_{\tau\in\widehat{K}} V(\tau)$ where $V(\tau)$ is the τ-isotypic component. In particular, a vector $v \in V$ has finite K-support, i.e., v is a finite sum of vectors $v_\tau \in V(\tau)$. There is a natural projection p_τ of V to $V(\tau)$, which could be realized by integration against the complex conjugation of characters χ_τ of τ over K. Now for any $v \in V$, integrate against

$$\overline{\chi_v} = \sum_{v_\tau\neq 0} \overline{\chi_\tau} \tag{2.1}$$

on K is a projection onto $U_v \cong \bigoplus_{v_\tau\neq 0} U(\tau)$ fixing v. We call it p_v.

For $(\mathfrak{g}, K)$-module U, define $\check{U}$ to be the subspace $\mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})_{K-\text{finite}}$ of all K-finite vectors in $\mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$. If U is admissible, then $\mathrm{Hom}_{\mathbb{C}}(U, V)_{K-\text{finite}} \cong \check{U} \otimes V$ for any vector space V and $(\check{U})\check{} \cong U$. If U is an irreducible $(\mathfrak{g}, K)$-module, $\mathrm{Hom}_{\mathfrak{g},K}(U, U) \cong \mathbb{C}$.

To prove the main result, we need following lemma.

Lemma 4. *Let U be an irreducible admissible $(\mathfrak{g}, K)$-module. Let V be a $(\mathfrak{g}, K)$-submodule in $U \otimes W$ where W is some vector space. Then there is a subspace U' of W such that $V = U \otimes U'$.*

Proof. Let $U' = \{ w \in W \mid U \otimes \mathbb{C}w \subset V \}$. It is a subspace of W and $U \otimes U' \subset V$. By quotient out of $U \otimes U'$ and viewing $V/(U \otimes U')$ as a submodule of $U \otimes W/U'$, we only have to prove that $V = 0$ if $U' = 0$.

Suppose that $V \neq 0$. Since $V = \bigoplus_{\tau\in\widehat{K}} V(\tau)$, there is a $\tau \in \widehat{K}$ such that the τ isotypic component $V(\tau) \neq 0$. In particular, there is some $0 \neq v \in V(\tau)$ such that $v = \sum_{i=1}^{s} u_i \otimes u'_i$ with $\{ u_i \}$ linearly independent and $u'_1 \neq 0$. Note that $\mathcal{U}(\mathfrak{g})^K$ and K act on the $U(\tau)$ isotypic component irreducibly since U is irreducible admissible. The subalgebra generated by $\mathcal{U}(\mathfrak{g})^K$ and K actions in $\mathrm{End}_{\mathbb{C}}(U(\tau))$ is the whole algebra (by Jacobson Density Theorem). In particular, there is a finite combination π of $\mathcal{U}(\mathfrak{g})^K$

[3]They proved the lemma in p-adic case. They only need a projection to the space of K-fixed vector. In our case, we have to project to K-isotypic component first.

[4]I learned the argument from Gordan Savin.

and K such that

$$\pi \cdot u_i = \begin{cases} u_1 & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

Hence $u_1 \otimes u_1' = \pi \cdot v \in V$. Since U is irreducible, u_1 generate U. Therefore $0 \neq u_1' \in U'$, which is contradict to $U' = 0$. □

Let V be a $(\mathfrak{g}, K)$-module and U be an irreducible admissible $(\mathfrak{g}, K)$-module. Define a $(\mathfrak{g}, K)$-module (c.f. Section 3.2),

$$\Omega_{V,U} = V/\mathcal{N}_{V,U}, \quad \text{where} \quad \mathcal{N}_{V,U} = \bigcap_{T \in \mathrm{Hom}_{\mathfrak{g},K}(V,U)} \mathrm{Ker}\,(T).$$

Lemma 5. *Then the maximal quotient* $\Omega_{V,U} \cong U \otimes U'$ *with* $U' \cong \mathrm{Hom}_{\mathfrak{g},K}(U, \Omega_{V,U})$. *Moreover,* $\mathrm{Hom}_{\mathfrak{g},K}(V, V)$ *act on* U' *by composition.*

Proof. Replacing V by $V/\mathcal{N}_{V,U}$, we can assume

$$\mathcal{N}_{V,U} = \bigcap_{T \in \mathrm{Hom}_{\mathfrak{g},K}(V,U)} \mathrm{Ker}\,(T) = 0,$$

i.e. $V = \Omega_{V,U}$. Let $\check{U} \cong \mathrm{Hom}(U, \mathbb{C})_{K-\text{finite}}$ be the dual of U in the category of $(\mathfrak{g}, K)$-modules. Let $W = (\check{U} \otimes V)_{\mathfrak{g},K}$ be the co-invariant of $(\mathfrak{g}, K)$ in $\check{U} \otimes V$, which is the maximal quotient $\Omega_{\check{U}\otimes V,\mathbb{C}}$ by definition. Let $p\colon \check{U} \times V \otimes \mathbb{C} \to (\check{U} \otimes V)_{\mathfrak{g},K} = W$ be the corresponding projection. Define $\phi\colon V \to \mathrm{Hom}_{\mathbb{C}}(\check{U}, W)$ by $v \mapsto (\check{u} \mapsto p(\check{u} \otimes v))$.

For any $v \in V$, let χ_v be the projection defined by (2.1). Now

$$\begin{aligned} \phi(v)(\check{u}) =& p(\check{u} \otimes v) = p\left(\check{u} \otimes \int_K \overline{\chi_v}(k) k \cdot v \, dk\right) \\ =& p\left((\int_K \overline{\chi_v}(k^{-1}) k \cdot \check{u} \, dk) \otimes v \right) \in W. \end{aligned}$$

Since $\chi_\tau(k^{-1})$ is the character of the dual τ^* of τ, $\phi(v)$ is in the space $\mathrm{Hom}_{\mathbb{C}}(\check{U}_v, W) \subset \mathrm{Hom}_{\mathbb{C}}(\check{U}, W)$[5]. Here $\check{U}_v = \bigoplus_{v_\tau \neq 0} U(\check{\tau})$ is finite dimension. So $\phi(v)$ is K-finite and ϕ factor through $U \otimes W \cong \mathrm{Hom}_{\mathbb{C}}(\check{U}, W)_{K-\text{finite}}$.

One the other hand, ϕ is injective. In fact, by assumption $\mathcal{N}_{V,U} = 0$, for each $0 \neq v \in V$, there is $T \in \mathrm{Hom}_{\mathfrak{g},K}(V, U)$ such that $T(v) \neq 0$. So there is a $\check{u} \in \check{U}$ such that $\check{u}(T(v)) \neq 0$. Notice that $f\colon \check{U} \otimes V \xrightarrow{\mathrm{id}\otimes T} \check{U} \otimes U \xrightarrow{\text{paring}} \mathbb{C}$ factor through W and let $\bar{f}\colon W \to \mathbb{C}$ satisfies $\bar{f} \circ p = \text{paring} \circ (\mathrm{id} \otimes T)$. We have $\phi(v) \neq 0$ since $\bar{f}(\phi(v)(\check{u})) \neq 0$.

Now we can view V as a $(\mathfrak{g}, K)$-submodule of $U \otimes W$ via ϕ. By Lemma 4,

[5] The inclusion is given by pre-composite with the projection onto $\check{U}_v$

$V = U \otimes U'$, where U' is for some subspace of W.

Now, $W = (\check{U} \otimes V)_{\mathfrak{g},K} \cong (\check{U} \otimes U \otimes U')_{\mathfrak{g},K} \cong U' \cong \mathrm{Hom}_{\mathfrak{g},K}(U, U \otimes U') \cong \mathrm{Hom}_{\mathfrak{g},K}(U, V)$. So we conclude that $V \cong U \otimes \mathrm{Hom}_{\mathfrak{g},K}(U, V)$. It is clear that $\mathrm{Hom}_{\mathfrak{g},K}(V, V)$ act on the second factor. □

2.3.4 Models of oscillator representation and $\mathcal{U}(\mathfrak{g})^H$-action

We will give some remarks on (Fock) models of oscillator representation following from Howe [How89a] and J. Adams' notes [Ada07], which is due to Steve Kudla. Due to these remarks, we will prove following Proposition.

Proposition 6. *Let (G, G') and (H, H') be a see-saw pair in $\mathrm{Sp}(W)$ such that $H \leq G$ and $G' \leq H'$. Let ω be an oscillator representation of $\widetilde{\mathrm{Sp}}(W)$, then as subalgebras of $\mathrm{End}_{\mathbb{C}}(\mathscr{Y})$,*

$$\omega(\mathcal{U}(\mathfrak{g})^{H_{\mathbb{C}}}) = \omega(\mathcal{U}(\mathfrak{g})^{H}) = \omega(\mathcal{U}(\mathfrak{h}')^{G'}) = \omega(\mathcal{U}(\mathfrak{h}')^{G'_{\mathbb{C}}}). \tag{2.2}$$

Moreover, there exist a map $\Xi\colon \mathcal{U}(\mathfrak{g})^{H_{\mathbb{C}}} \to \mathcal{U}(\mathfrak{h}')^{G'_{\mathbb{C}}}$ (independent of real forms, may not unique and not be algebra homomorphism) such that $\omega(x) = \omega(\Xi(x))$.

Remark:

1. The above proposition provides a tool to translate the Hecke-algebra, $\mathcal{U}(\mathfrak{g})^H$, actions from one side to the other side in see-saw pair. We will use this proposition to study the derived functor modules of theta lifts in Chapter 3.

2. If $(H, H') = (G, G')$, Proposition 44 will implies the well know formula $\mathcal{Z}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}')$, which will lead the correspondence of infinitesimal characters, see [Prz96].

3. In Lee, Nishiyama and Wachi's paper [LNW08], they also observed (3.6) and use it to study a generalization of Capelli identity.

Let $W_{\mathbb{C}}$ be a complex symplectic space with non-degenerate symplectic form $\langle\ ,\ \rangle$. Fix a non-trivial character ψ of $\mathbb{C}$ where view $\mathbb{C}$ as an abelian complex Lie algebra. More precisely, $\psi(z) = \lambda z$ for some $\lambda \in \mathbb{C}^{\times}$.

Define

$$\Omega(W_{\mathbb{C}}) := \mathcal{T}(W_{\mathbb{C}})/\mathcal{I}$$

where

$$\mathcal{T}(W_{\mathbb{C}}) = \bigoplus_{j \in \mathbb{N}} \underbrace{W_{\mathbb{C}} \otimes \cdots \otimes W_{\mathbb{C}}}_{j \text{ times}}$$

is the tensor algebra of $W_{\mathbb{C}}$ and $\mathcal{I}$ is the two side ideal in $\mathcal{T}(W_{\mathbb{C}})$ generated by

$$\{ v \otimes w - w \otimes v - \psi(\langle v, w \rangle) \} .$$

$\Omega(W_{\mathbb{C}})$ has a natural filtration induced by the natural filtration on $\mathcal{T}(W_{\mathbb{C}})$. Let $\Omega_j(W_{\mathbb{C}})$ be the space of elements of degree less and equal to j and $\Omega^j(W_{\mathbb{C}})$ be image of $\mathcal{T}^j(W_{\mathbb{C}})$. The corresponding graded algebra of $\Omega(W_{\mathbb{C}})$ will isomorphic to $\mathbb{C}[W_{\mathbb{C}}]$. Let $\mathfrak{e} = W_{\mathbb{C}} \oplus L$ be the Heisenberg Lie algebra of $W_{\mathbb{C}}$, where $L \cong \mathbb{C}\mathbb{1}$ is the center of $\mathfrak{h}$. Let $[v, w] = vw - wv$ be the *commutator* and $\{v, w\} = vw + wv$ be the *anti-commutator*. Now $[v, w] = \langle v, w \rangle \mathbb{1}$ for any $v, w \in W_{\mathbb{C}}$. The complex symplectic group $\mathrm{Sp}(W_{\mathbb{C}})$ has a natural action on $\mathcal{T}(W_{\mathbb{C}})$ and therefore induce an action on $\Omega(W_{\mathbb{C}})$. Let $\mathfrak{sp} = \mathfrak{sp}(W_{\mathbb{C}})$ be the complex Lie algebra of $\mathrm{Sp}(W_{\mathbb{C}})$.

Lemma 7 ([Ada07, Section 2]). *(i)* $\Omega(W_{\mathbb{C}}) \cong \mathcal{U}(\mathfrak{e})/\langle \mathbb{1} - \psi(1) \rangle$*;*

(ii) $\Omega_1(W_{\mathbb{C}}) \cong \mathfrak{e}$*;*

(iii) $\Omega_1(W_{\mathbb{C}})/\Omega_0(W_{\mathbb{C}}) \cong W_{\mathbb{C}}$*;*

(iv) $\Omega_2(W_{\mathbb{C}})/\Omega_1(W_{\mathbb{C}}) \cong \mathcal{S}^2(W_{\mathbb{C}}) \cong \mathfrak{sp}(W_{\mathbb{C}})$ *via the action of* $\Omega^2(W_{\mathbb{C}})$ *on* $\Omega^1(W_{\mathbb{C}})/\Omega^0(W_{\mathbb{C}}) \cong W_{\mathbb{C}}$*;*
More precisely, $\mathfrak{sp}(W_{\mathbb{C}}) \ni x = \{a, b\} \in \mathcal{S}^2(W_{\mathbb{C}})$ *act on* $c \in W_{\mathbb{C}} \cong \Omega^1(W_{\mathbb{C}})$ *by* $x(c) = [\{a, b\}, c]/\lambda$*;*

(v) $\Omega_2(W_{\mathbb{C}}) \cong \mathfrak{sp} \ltimes \mathfrak{h}$ *is a semi-direct product of Lie algebra.*

Fix a complex polarization of $W_{\mathbb{C}}$, i.e. a decomposition $W_{\mathbb{C}} = X \oplus Y$ such that X and Y are maximal isotropic subspaces in $W_{\mathbb{C}}$. Define

$$\mathscr{Y} := \Omega(W_{\mathbb{C}})/\Omega(W_{\mathbb{C}})X. \tag{2.3}$$

Since $Y \cong X^*$,

$$\mathscr{Y} \cong \mathcal{S}(Y) \cong \mathcal{S}[X^*] \cong \mathbb{C}[X].$$

Let $\Omega(W_{\mathbb{C}})$ act on $\mathscr{Y} \cong \Omega(W_{\mathbb{C}})/\Omega(W_{\mathbb{C}})X$ by left multiplication. One can verify that, Y act on $\mathscr{Y}$ by multiplying linear polynomials, and X act on $\mathscr{Y}$ by differentiation. So $\Omega(W_{\mathbb{C}})$ act on $\mathscr{Y}$ as polynomial coefficients differential operators. Moreover, $\Omega(W_{\mathbb{C}})$ act irreducibly and faithfully on $\mathscr{Y}$. By this representation, $\Omega(W_{\mathbb{C}})$ isomorphic to as subalgebra (Weyl algebra) $\mathrm{End}^\circ \subset \mathrm{End}_{\mathbb{C}}(\mathscr{Y})$ as in [How89a]. The inclusion $\mathfrak{sp} = \mathcal{S}^2(W_{\mathbb{C}}) \subset \Omega_2(W_{\mathbb{C}})$ induces map

$$\omega_{\mathbb{C}} \colon \mathcal{U}(\mathfrak{sp}) \to \Omega(W_{\mathbb{C}}) \cong \mathrm{End}^\circ.$$

Therefore $\omega_{\mathbb{C}}$ is a representation of $\mathcal{U}(\mathfrak{sp})$ on $\mathscr{Y}$. In fact, it will realize the Fock module of the oscillator representation (as the notation already suggested). Keep in mind that the Lie algebra $\mathfrak{sp}$ has following decomposition into Lie subalgebras:

$$\mathfrak{sp} = \mathcal{S}^2(W_{\mathbb{C}}) = X \otimes Y \otimes \oplus S^2(Y) \oplus S^2(X).$$

In Howe's notation [How89b], $\mathfrak{sp}^{(1,1)} := X \otimes Y$, $\mathfrak{sp}^{(2,0)} := S^2(Y)$ and $\mathfrak{sp}^{(0,2)} := S^2(X)$.

Let $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ be a complex dual pair in $\mathrm{Sp}(W_{\mathbb{C}})$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are naturally embedded in $\mathfrak{sp} \subset \Omega^2(W_{\mathbb{C}})$. Moreover, by classical invariant theory [How89a],

$$\omega_{\mathbb{C}}(\mathcal{U}(\mathfrak{g})) = \Omega(W_{\mathbb{C}})^{G'_{\mathbb{C}}} \quad \text{and} \quad \omega_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}')) = \Omega(W_{\mathbb{C}})^{G_{\mathbb{C}}}. \tag{2.4}$$

From now on, we will take $\psi(z) = \lambda z$ with $\lambda = \sqrt{-1}$. Let W be a real symplectic subspace of $W_{\mathbb{C}}$ such that $(W)_{\mathbb{C}} = W_{\mathbb{C}}$ and the symplectic form $\langle\ ,\ \rangle$ restricted on W is non-degenerate. Fix a complex polarization $W_{\mathbb{C}} = X \oplus Y$. It called *totally complex polarization* [Ada07] if $X \cap W = 0$. This is equivalent to choose a complex structure $J \in \mathfrak{sp}(W)$ on W (so $J^2 = -\mathrm{id}$ and J is the operator of multiplication by i). We associate a non-degenerate Hermitian form $(\ ,\ )$ on W, such that $\langle v, w\rangle = \mathrm{Im}\,(v, w)$, i.e. $(v, w) = \langle Jv, w\rangle + i\,\langle v, w\rangle$. Extend J to $W_{\mathbb{C}} = (W)_{\mathbb{C}}$ linearly, X will be the i-eigen space of J and Y will be the $-i$-eigen space. By the definition of totally complex polarization, we have

$$X \oplus Y = W_{\mathbb{C}} = W \oplus iW,$$

and the projection to W gives an $\mathbb{R}$-linear isomorphism $X \to W$, one can directly check that this map is $\mathbb{C}$-linear if we view W as *complex vector space* $W^{\mathbb{C}}$ with structure J.

Now let $\mathfrak{u} := X \otimes Y$ and $\mathfrak{u}_0 := \mathfrak{u} \cap \mathfrak{sp}(W)$. Then $\mathfrak{u} \cong \mathfrak{gl}(W^{\mathbb{C}})$ is the complex Lie algebra of the general linear group of complex vector space $W^{\mathbb{C}}$ and $\mathfrak{u}(W^{\mathbb{C}})$ is the real Lie algebra of unitary group $\mathrm{U}(W^{\mathbb{C}})$ preserving form $(\ ,\ )$. In fact, $\mathfrak{u} = \mathfrak{sp}^J$ is the set of elements in $\mathfrak{sp}$ which commute with J. So for any $x \in \mathfrak{u}(W^{\mathbb{C}})$, $(xv, w) + (v, xw) = \langle Jxv, w\rangle + i\,\langle xv, w\rangle + \langle Jv, xw\rangle + i\,\langle v, xw\rangle = 0$.

When $(\ ,\ )$ is positive definite, $\mathfrak{u}_0$ is the Lie algebra of the maximal compact subgroup $\widetilde{\mathrm{U}}(W^{\mathbb{C}})$ in $\widetilde{\mathrm{Sp}}(W)$ and $\mathscr{Y}$ will be the Fock model of the oscillator representation of $\widetilde{\mathrm{Sp}}(W)$ attached to the unitary character

$\mathbb{R} \ni x \mapsto e^{\lambda x}$. Moreover, $Y \otimes X \cap \mathfrak{sp}(W) \cong \mathfrak{u}(W^{\mathbb{C}})$ is the Lie algebre of corresponding maximal compact subgroup.

Following lemma is a rephrase of the equation (2.4) in [How89b] and one can check it case by case according to the classification of irreducible reductive dual pairs. We omit the proof, but give some examples in the end of this section.

Lemma 8. *Fix a complex dual pair $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ in $\mathrm{Sp}(W_{\mathbb{C}})$. For every real form (G, G'), there is a real form W of $W_{\mathbb{C}}$ such that $(\ ,\ )$ is positive definite on $W^{\mathbb{C}}$ and*

$$\mathfrak{g}_0 = \mathfrak{sp}(W) \cap \mathfrak{g}, \quad \mathfrak{g}'_0 = \mathfrak{sp}(W) \cap \mathfrak{g}',$$

where $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{g}_0$, $\mathfrak{g}'_0$ and $\mathfrak{sp}(W)$ are view as Lie subalgebras of $\mathfrak{sp}(W_{\mathbb{C}})$. □

Proof of Proposition 6. By classical invariant theorem (2.4),

$$\omega_{\mathbb{C}}(\mathcal{U}(\mathfrak{g})) = \Omega(W_{\mathbb{C}})^{G'_{\mathbb{C}}}.$$

Note that all groups act on $\mathcal{U}(\mathfrak{sp})$ and $\Omega(W_{\mathbb{C}})$ reductively. So

$$\omega_{\mathbb{C}}(\mathcal{U}(\mathfrak{g})^{H_{\mathbb{C}}}) = \omega_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}))^{H_{\mathbb{C}}} = (\Omega(W_{\mathbb{C}})^{G'_{\mathbb{C}}})^{H_{\mathbb{C}}} = (\Omega(W_{\mathbb{C}})^{H_{\mathbb{C}}})^{G'_{\mathbb{C}}} = \omega_{\mathbb{C}}(\mathcal{U}(\mathfrak{m}')^{G'_{\mathbb{C}}}).$$

For every real form (G, G') of $(G_{\mathbb{C}}, G'_{\mathbb{C}})$, it is clear that

$$\mathcal{U}(\mathfrak{g})^{H} = \mathcal{U}(\mathfrak{g})^{H_{\mathbb{C}}}$$

by the classification of irreducible reductive dual pairs. Since oscillator representation ω of $\mathfrak{sp}(W)$ on the Fock space $\mathscr{Y}$ (c.f. (2.3)) factor through $\omega_{\mathbb{C}}$ (see following diagram), the choice of $\Xi(x)$ could be made independent of real forms via $\omega_{\mathbb{C}}$.

$$\begin{array}{ccccc}
\mathfrak{g} \oplus \mathfrak{g}' & \hookrightarrow & \mathfrak{sp}(W_{\mathbb{C}}) & \xrightarrow{\omega_{\mathbb{C}}} & \Omega(W_{\mathbb{C}}) \\
\uparrow & & \uparrow & \searrow^{\omega} & \downarrow \text{left multiplication} \\
\mathfrak{g}_0 \oplus \mathfrak{g}'_0 & \hookrightarrow & \mathfrak{sp}(W) & \xrightarrow{\omega} & \mathrm{End}_{\mathbb{C}}(\mathscr{Y})
\end{array}$$

□

In the rest of this section, we give an explicit construction of W for different real form of pair $(\mathrm{O}(m, \mathbb{C}), \mathrm{Sp}(2n, \mathbb{C}))$ appeared in Section 3.5.

Let $U \cong \mathbb{C}^m$ be a complex symmetric space with orthonormal basis $\{ a_i \}$ and $V \cong \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ be a complex symplectic space with symplectic basis $\{ b_i, c_i \}$ where b_i span a maximal isotropic subspace and c_i are the corresponding dual vectors. Let $W_{\mathbb{C}} = U \otimes V$. $\mathfrak{g} = \mathfrak{so}(U)$ and $\mathfrak{g}' = \mathfrak{sp}(V)$ be the subalgebra of $\mathfrak{sp}(W_{\mathbb{C}})$.

The map $\iota\colon \mathfrak{g} \to \mathfrak{sp}$ is given by

$$\begin{array}{rlcrl} \mathfrak{g} = & \bigwedge^2(U) & \to & \mathfrak{sp} = & \mathcal{S}^2(U \otimes V) \\ & [u_1, u_2] & \mapsto & & \sum_{j=1}^{n} \{u_1 \otimes b_j, u_2 \otimes c_j\}. \end{array}$$

The map $\iota'\colon \mathfrak{g}' \to \mathfrak{sp}$ is given by

$$\begin{array}{rlcrl} \mathfrak{g}' = & \mathcal{S}^2(V) & \to & \mathfrak{sp} = & \mathcal{S}^2(U \otimes V) \\ & \{v_1, v_2\} & \mapsto & & \sum_{j=1}^{m} \{a_i \otimes v_1, a_i \otimes v_2\}. \end{array}$$

For $p, q, r, s \in \mathbb{N}$ such that $p + q = \dim U = m$ and $2(r + s) = \dim V$, let

$$u_j = \begin{cases} a_j & j \leq p \\ ia_j & j > p \end{cases}, \quad e_j = \frac{1}{\sqrt{2}} \begin{cases} b_j - ic_j & j \leq r \\ ib_j + c_j & j > r \end{cases}, \quad f_j = \frac{1}{\sqrt{2}} \begin{cases} -ib_j + c_j & j \leq r \\ b_j + ic_j & j > r \end{cases},$$

$$U_0 = \mathrm{span}_{\mathbb{R}}\{\, \mathrm{u_i} \,\}, \qquad V_0 = \mathrm{span}_{\mathbb{R}}\{\, \mathrm{e_j}, \mathrm{f_j} \,\}$$

and $W = U_0 \otimes V_0 = \mathrm{span}_{\mathbb{R}}\{\, \mathrm{u_i} \otimes \mathrm{e_j}, \mathrm{u_i} \otimes \mathrm{f_j} \,\}$.

By definition, $\{\, u_i \otimes e_j, u_i \otimes f_j \,\}$ form a symplectic basis of W. Define a complex structure J on W by

$$J(u_i \otimes e_j) = -u_i \otimes f_j \quad \text{and} \quad J(u_j \otimes f_j) = u_i \otimes e_j.$$

Then the i-eigenspace is $X = \mathrm{span}\{\, \mathrm{a_i} \otimes \mathrm{b_j} \,\}$ and $-i$-eigenspace space is $Y = \mathrm{span}\{\, \mathrm{a_i} \otimes \mathrm{c_j} \,\}$. Denote $\mathfrak{u} = X \otimes Y \subset \mathfrak{sp}(W_{\mathbb{C}})$. Then

$$\begin{aligned} \mathfrak{g} \cap \mathfrak{sp}(W) &= \mathfrak{so}(p, q), & \mathfrak{g}' \cap \mathfrak{sp}(W) &= \mathfrak{sp}(2n), \\ \mathfrak{u} \cap \mathfrak{g} \cap \mathfrak{sp}(W) &= \mathfrak{so}(p) \oplus \mathfrak{so}(q), & \mathfrak{u} \cap \mathfrak{g}' \cap \mathfrak{sp}(W) &= \mathfrak{u}(r, s). \end{aligned}$$

On the other hand, define another complex structure J_c on W by

$$J_c(u_i \otimes e_j) = \begin{cases} -u_i \otimes f_j & j \leq r \\ u_i \otimes e_j & j > r \end{cases}, \quad J_c(u_i \otimes f_j) = \begin{cases} u_i \otimes e_j & j \leq r \\ u_i \otimes f_j & j > r \end{cases}.$$

Then

$$X_c = \text{span}\{\, \mathrm{a_i} \otimes \mathrm{b_j} \mid \mathrm{j} \le \mathrm{r} \,\} \cup \{\, \mathrm{a_i} \otimes \mathrm{c_j} \mid \mathrm{j} > \mathrm{r} \,\} \quad \text{and}$$
$$Y_c = \text{span}\{\, \mathrm{a_i} \otimes \mathrm{c_j} \mid \mathrm{j} \le \mathrm{r} \,\} \cup \{\, \mathrm{a_i} \otimes \mathrm{b_j} \mid \mathrm{j} > \mathrm{r} \,\}.$$

The corresponding form $(\ ,\ )_c$ on $W^{\mathbb{C}}$ is positive definite and $\mathfrak{u}_c = X_c \otimes Y_c$ is the complexification of the Lie algebra of a maximal compact subgroup of $\mathrm{Sp}(W)$.

In Chapter 3, we will study two real forms in a complex group simultaneously. We will choose two real forms W_1 and W_2 of $W_{\mathbb{C}}$. Then define $\mathfrak{g}_j = \mathfrak{g} \cap \mathfrak{sp}(W_j)$ and $\mathfrak{g}'_j = \mathfrak{g} \cap \mathfrak{sp}(W_j)$ for $j = 1, 2$. We also will choose $\mathfrak{u}_j$ such that $\mathfrak{g}_j \cap \mathfrak{u}_j$ is a maximal compact Lie subalgebra of $\mathfrak{g}_j$.

In Section 3.5.1, we will let:

- W_1 is defined by $p = m, q = 0, r = r, s = s$. Let $\mathfrak{u}_1 = \mathfrak{u}_c$.
- W_2 is defined by $p = p, q = q, r = n, s = 0$. Let $\mathfrak{u}_2 = \mathfrak{u}$.

In Section 3.5.2 we will let:

- W_1 is defined by $p = p, q = q, r = n, s = 0$. Let $\mathfrak{u}_1 = \mathfrak{u} = \mathfrak{u}_c$.
- W_2 is defined by $p = p + r, q = q - r, r = n, s = 0$. Let $\mathfrak{u}_2 = \mathfrak{u} = \mathfrak{u}_c$.

2.3.5 Compact dual pairs

Now we will summarize some well known facts about compact dual pairs and their relationship with classical invariant theory. All these results could be found in Howe's work [How89a] [How95] and is fundamental for local theta correspondence over $\mathbb{R}$.

A real reductive dual pair (G, G') is called a *compact dual pair*, if one of G is compact. We list all irreducible compact dual pairs over $\mathbb{R}$ in Table 2.3. Here n_2 or n_1 could be 0, which is the only case that both G and G' are compact.

	G	$G' = K'$	$K_{\mathbb{C}}$	$G'_{\mathbb{C}} = K'_{\mathbb{C}}$
Case $\mathbb{R}$	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{O}(m)$	$\mathrm{GL}(n, \mathbb{C})$	$\mathrm{O}(m, \mathbb{C})$
Case $\mathbb{C}$	$\mathrm{U}(n_1, n_2)$	$\mathrm{U}(m)$	$\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C})$	$\mathrm{GL}(m, \mathbb{C})$
Case $\mathbb{H}$	$\mathrm{O}^*(2n)$	$\mathrm{Sp}(m)$	$\mathrm{GL}(n, \mathbb{C})$	$\mathrm{Sp}(m, \mathbb{C})$

Table 2.3: Compact dual pairs

2.3.5.1 Parametrization of irreducible modules

We adopt the usual convention to parametrize irreducible representations of the compact classical groups (c.f. [How95] or [GW09]).

Write τ_G^μ for the element in $\widehat{G}$ corresponding to parameter μ where G could be a compact group in Table 2.3, its double covering or its complexification[6].

$\widehat{\mathrm{U}(m)}$ is parametrized by arrays of integers

$$(a_1, \cdots, a_m),$$

where a_i are non-increasing strings of integers (may be negative). Fixing a standard root system of $\mathrm{U}(m)$, for such array μ, $\tau_{\mathrm{U}(m)}^\mu$ denote the irreducible $\mathrm{U}(m)$-module with highest weight μ.

Since $\mathrm{O}(m)$ and $\mathrm{Sp}(2m)$ are subgroups of $\mathrm{U}(m)$ and $\mathrm{U}(2m)$ their irreducible representations can be constructed by restriction.

$\widehat{\mathrm{O}(m)}$ is parametrized by non-increasing arrays of integers

$$(a_1, \cdots, a_k, \underbrace{\epsilon, \cdots \epsilon}_{m-2k}, \underbrace{0, \cdots, 0}_{k})$$

where $2k \leq m$ and $a_k > 0$ (resp. $a_k \geq 2$) if $\epsilon = 0$ (resp. $= 1$). $\tau_{\mathrm{O}(m)}^\mu$ denote the irreducible $\mathrm{O}(m)$-module generated by the highest weight vector in $\tau_{\mathrm{U}(m)}^\mu$.

$\widehat{\mathrm{Sp}(m)}$ is parametrized by non-increasing arrays of non-negative integer

$$(a_1, \cdots, a_m)$$

which corresponding to the highest weight under standard basis. $\tau_{\mathrm{Sp}(m)}^\mu$ denote the irreducible $\mathrm{Sp}(m)$-module generated by the highest weight vector in $\tau_{\mathrm{U}(2m)}^{(a_1,\cdots,a_m,0,\cdots 0)}$.

Note that the double covering $\widetilde{G}$ is depends on dual pairs. The maximal compact subgroup $\widetilde{\mathrm{U}}(N)$ in the metaplectic group $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$ and the double

[6] In this case, it means an irreducible holomorphic representation

cover $\widetilde{G}$ of a compact subgroup G in $\mathrm{U}(N)$ are given by following pullback.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{G} & \xrightarrow{\pi} & G & \longrightarrow & 0\,, \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{\mathrm{U}}(N) & \longrightarrow & \mathrm{U}(N) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \det} & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{C}^\times & \xrightarrow{s} & \mathbb{C}^\times & \longrightarrow & 0
\end{array}$$

where $s\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is the square map $z \mapsto z^2$ and $\mathbb{Z}/2\mathbb{Z}$ is the center of $\widetilde{\mathrm{Sp}}(N, 2\mathbb{R})$. Note that $\widetilde{G} \subset \mathbb{C}^\times \times G$, we always choose the projection onto $\mathbb{C}^\times$ to be the fixed genuine character of $\widetilde{G}$ later.

For dual pair $(\mathrm{U}(n), \mathrm{U}(m))$ in $\mathrm{Sp}(2nm, \mathbb{R})$, $\widetilde{\mathrm{U}}(m)$ is isomorphic to following pullback

$$\begin{array}{ccc}
\widetilde{\mathrm{U}}(m) & \longrightarrow & \mathrm{U}(m) \quad ; \\
\downarrow & & \downarrow{\scriptstyle \det^n} \\
\mathbb{C}^\times & \xrightarrow{s} & \mathbb{C}^\times
\end{array}$$

and so, $\widetilde{\mathrm{U}}(m) \cong \mathbb{Z}/2\mathbb{Z} \times \mathrm{U}(m)$ if n is even; $\widetilde{\mathrm{U}}(m)$ is a connected double cover if n is odd. Moreover $\widetilde{\mathrm{U}}(m)$ are isomorphism for n with same parity. In all cases, to describe the genuine representation of $\widetilde{\mathrm{U}}(m)$ it is enough describe the action of $\mathfrak{u}(m)$. So, we aslo parametrize irreducible representations of $\widetilde{\mathrm{U}}(m)$ by heights weights: it is array of non-increasing integers (resp. half-integer) with length m, if n is even (resp. odd).

Genuine irreducible representations of $\widetilde{\mathrm{O}}(m)$(resp. $\widetilde{\mathrm{Sp}}(m)$) are also constructed by restricting to irreducible modules generated by highest weight vectors of irreducible $\widetilde{\mathrm{U}}(m)$-modules (resp. $\widetilde{\mathrm{U}}(2m)$-module). But, as convention, define a map $\widehat{\mathrm{O}(m)} \to \widehat{\widetilde{\mathrm{O}}(m)}$ by $\tau^\mu_{\mathrm{O}(m)} \mapsto (\tau^\mu_{\mathrm{O}(m)} \circ \pi) \otimes \varsigma$ and identify genuine $\widetilde{\mathrm{O}}(m)$-module with $\mathrm{O}(m)$-module, where ς is a fixed character and $\pi\colon \widetilde{\mathrm{O}}(m) \to \mathrm{O}(m)$ is the natural projection. Similarly identify $\widetilde{\mathrm{Sp}}(m)$-module with $\mathrm{Sp}(m)$-module.

We identify locally finite representations of compact groups with rational representations of their complexifications. Furthermore, we will use same notation to indicate the representations of Lie algebras of these groups. An array of integers will also identify with *Young digram* such that the n-th entry is the length of n-th row in the diagram.

In general, for any subgroup G in Sp, we will identify genuine representation of $\widetilde{G}$ with G-module by twisting with certain genuine character of $\widetilde{G}$

(if it exists[7]).

2.3.5.2 Explicit decomposition for compact dual pairs

Now we can state the well known theorem says that $\mathscr{Y}$ decompose into direct sums under compact dual pair actions.

Theorem 9 ([KV78] [How89a]). *Let (G, G') be a compact dual pair with G' compact. Let K be the maximal compact subgroup of G. Then*

$$\mathscr{Y} = \bigoplus_{\tau^{\mu}_{\widetilde{G'}} \in \mathcal{R}(\widetilde{G'};\mathscr{Y})} L_{\widetilde{G}}(\mu) \otimes \tau^{\mu}_{\widetilde{G'}}$$

where $\tau^{\mu}_{\widetilde{G'}}$ run over the set of irreducible $\widetilde{G'}$-modules occur in $\mathcal{R}(\widetilde{G'};\mathscr{Y})$ and $L_{\widetilde{G}}(\mu)$ is the irreducible unitary lowest weight $(\mathfrak{g}, \widetilde{K})$-module corresponding to parameter μ.

The explicit descriptions are as following.

(i) For dual pair $(\mathrm{U}(r,s), \mathrm{U}(m))$ in $\mathrm{Sp}(2(r+s)m, \mathbb{R})$,

$$\mathscr{Y} = \bigoplus_{\mu} L_{\widetilde{\mathrm{U}}(r,s)}(\mu) \otimes \tau^{\mu + \frac{r-s}{2}\mathbf{1}_m}_{\widetilde{\mathrm{U}}(m)}.$$

Here μ run over the set of arrays $(a_1, \cdots, a_k, 0, \cdots, 0, -b_l, \cdots, -b_1)$ such that $k \leq r$, $l \leq s$ and a_j, b_j are non-increasing string of positive integers and zeros between a_k and $-b_l$ are added if necessary to make μ of length m. $L_{\widetilde{\mathrm{U}}(r,s)}(\mu)$ is the irreducible unitary lowest weight $(\mathfrak{gl}(r+s, \mathbb{C}), \widetilde{\mathrm{U}}(r) \times \widetilde{\mathrm{U}}(s))$-module with lowest $\widetilde{\mathrm{U}}(r) \times \widetilde{\mathrm{U}}(s)$-type

$$\tau^{(a_1,\cdots,a_k,0,\cdots,0)+\frac{m}{2}}_{\widetilde{\mathrm{U}}(r)} \otimes \left(\tau^{(b_1,\cdots,b_l,0,\cdots,0)+\frac{m}{2}}_{\widetilde{\mathrm{U}}(s)}\right)^{*}.$$

(ii) For dual pair $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(m))$ in $\mathrm{Sp}(2nm, \mathbb{R})$,

$$\mathscr{Y} = \bigoplus_{\mu} L_{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(\mu) \otimes \tau^{\mu}_{O(m)}.$$

Here μ run over the set of all arrays with length less than $\min(n, m)$ such that $\tau^{\mu}_{\mathrm{O}(m)}$ make sense. $L_{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(\mu)$ is the irreducible lowest weight $(\mathfrak{sp}(2n, \mathbb{C}), \widetilde{\mathrm{U}}(m))$-module with lowest $\widetilde{\mathrm{U}}(m)$-type $\tau^{\mu+\frac{m}{2}\mathbf{1}_n}_{\widetilde{\mathrm{U}}(n)}$. Moreover, the above description fix the choice of the genuine character of $\widetilde{\mathrm{O}}(m)$ implicitly.

[7]In fact, except for $\mathrm{Sp}(2n, \mathbb{R})$ in dual pair $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$ such that $p+q$ is odd, genuine character(s) always exist

(iii) For dual pair $(\mathrm{O}^*(2n), \mathrm{Sp}(m))$ *in* $\mathrm{Sp}(4nm, \mathbb{R})$,

$$\mathscr{Y} = \bigoplus_{\mu} L_{\widetilde{\mathrm{O}}^*(2n)}(\mu) \otimes \tau^{\mu}_{\mathrm{Sp}(m)}.$$

Here μ *run over the set of all decreasing non-negative integers with length less than* $\min\{n, m\}$. $L_{\widetilde{O}^*(2n)}$ *is the irreducible lowest weight module of* $(\mathfrak{so}(2n, \mathbb{C}), \widetilde{U}(n))$ *with lowest* $\widetilde{U}(n)$*-type* $\tau^{\mu+m\mathbf{1}_n}_{\widetilde{U}(n)}$. *Since* $\widetilde{\mathrm{Sp}}(m)$ *isomorphic to* $\mathrm{Sp}(m)$ *direct product with the center, there is a unique genuine character of* $\widetilde{\mathrm{Sp}}(m)$ *and above description will not cause ambiguity.*

2.3.6 Theta lifts of characters

We are interested in the theta lifts of one dimensional representations. Although they are simple, the study of these representations can lead deep results. In this section, let ρ be a genuine character of $\widetilde{G}$, we will give some properties of its (full) theta lift.

We still adopt Howe's notation [How89b] and let M' be the subgroup in Sp such that (K, M') is a compact dual pair and M' is Hermitian symmetric.

Lemma 10. *The maximal Howe quotient* $\Theta(\rho)$ *is* $\widetilde{K}$*-multiplicity free for any character* $\rho \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$. *Moreover,* $\Theta(\rho')$ *is isomorphic to the Harish-Chandra module (space of* $\widetilde{K}'$*-finite vectors) of* $\Theta^\infty(\rho')$.

Proof. For any $\widetilde{K}$-type τ occur in $\Theta(\rho)$, $L(\tau') := \Theta(\tau)$ is a lowest weight $(\mathfrak{m}', \widetilde{M}'^{(1,1)})$-module with lowest $\widetilde{M}'^{(1,1)}$-type τ' determined by τ. Let $\mathfrak{q}' = \mathfrak{m}'^{(1,1)} \oplus \mathfrak{m}'^{(0,2)}$. Then $L(\tau')$ is a quotient of the generalized Verma module $V(\tau') = \mathcal{U}(\mathfrak{m}') \otimes_{\mathfrak{q}'} \tau'$. On the other hand $\mathfrak{m}' = \mathfrak{m}'^{(2,0)} + \mathfrak{q}' = \mathfrak{g}' + \mathfrak{q}'$ and $\mathfrak{k}' = \mathfrak{g}' \cap \mathfrak{q}'$. Therefore $V(\tau') \cong \mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{k}')} \tau'$ as $(\mathfrak{g}', \widetilde{K}')$-module. Now, by a see-saw pair argument,

$$\begin{aligned}
\dim \mathrm{Hom}_{\widetilde{K}}(\Theta(\rho), \tau) &= \dim \mathrm{Hom}_{\mathfrak{g}', \widetilde{K}'}(L(\tau'), \rho) \\
&\leq \dim \mathrm{Hom}_{\mathfrak{g}', \widetilde{K}'}(V(\tau'), \rho) = \dim \mathrm{Hom}_{\mathfrak{g}', \widetilde{K}'}(\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{k}')} \tau', \rho) \\
&= \dim \mathrm{Hom}_{\widetilde{K}'}(\tau', \rho)
\end{aligned} \tag{2.5}$$

By the structure of dual pairs, $\mathfrak{m}'^{(1,1)}$ always is a product of unitary Lie algebra and one can view $(\mathfrak{m}'^{(1,1)}, \widetilde{K}')$ as a Harish-Chandra pair with $\widetilde{K}'$ maximal compact. Therefore the lemma follows since $\dim \mathrm{Hom}_{\widetilde{K}'}(\tau', \rho) \leq \dim \rho = 1$

The second claim hold by apply following automatic continuity theorem to the pair M' and G'. □

This theorem is due to van den Ban-Delorme and Brylinski-Delorme[BD92] and I learned it from Sun [Sun11].

Theorem 11. *Let G be a real reductive group and θ be a Cartan involution respect to maximal compact subgroup K. Let σ be a involution on G commute with θ. Let H be an open subgroup of the σ-fixed point group G^σ. Now $K_H = H \cap K$ is a maximal compact subgroup of H. Let E be a finitely generated admissible $(\mathfrak{g}, K)$-module and $\rho\colon H \to \mathbb{C}^\times$ be a character of H. Then the restriction induces a linear isomorphism*

$$\mathrm{Hom}_H(E^\infty, \rho) \cong \mathrm{Hom}_{\mathfrak{h}, K_H}(E, \rho)$$

where E^∞ denote the Casselman-Wallach globalization of E.

The next lemma is essentially from Huang and Zhu [ZH97].

Lemma 12. *Let (G, G') be a type-I dual pair in the stable range such that G' is the smaller group (except for $(G, G') = (\mathrm{O}(2n, 2n), \mathrm{Sp}(2n, \mathbb{R}))$. Then for every genuine unitary character ρ of $\widetilde{G}'$, $\Theta(\rho) = \theta(\rho)$ is irreducible and unitarizable.*

Proof. Clearly, we only need to prove $\Theta(\rho) = \theta(\rho)$ since the irreducibility and unitarity of $\theta(\rho)$ is know by[Li89] (the ideas could be at least trace back to the late 1970s [How79a] [How80]). We will prove that a $\widetilde{K}$-type τ occur in $\theta(\rho)$ if and only if $\mathrm{Hom}_{\widetilde{K}'}(\tau', \rho) \neq 0$, and then, $\Theta(\rho)$ is the $(\mathfrak{g}, \widetilde{K})$-module of $\theta^\infty(\rho)$ by (2.5) and its multiplicity freeness. The "only if" part is from the proof of Lemma 10. The proof of "if" part is from [Li90] and [NZ04].

Since $\rho\colon \widetilde{G}' \to \mathbb{C}^\times$ is an unitary character, Li's construction of $\theta(\rho)$ in stable range defines a $\widetilde{G}$ invariant form on $\mathscr{Y}^\infty$ as following:

$$(\Phi_1, \Phi_2)_\rho = \int_{\widetilde{G}'} (\Phi_1, \overline{\rho}(g)\omega(g)\Phi_2)\, dg \quad \forall \Phi_1, \Phi_2 \in \mathscr{Y}^\infty$$

where $(\ ,\ )$ denote the Hermitian inner product on $\mathscr{Y}^\infty$. The above integration is well defined by the stable range condition (Corollary 3.3 [Li89]). Let $\mathcal{R}_\rho$ be the radical of form $(\ ,\ )_\rho$. Then $\theta^\infty(\rho) \cong \mathscr{Y}^\infty/\mathcal{R}_\rho$is non-zero and irreducible. Moreover, $\theta^\infty(\rho)$ is unitary under the form $(\ ,\ )_\rho$.

Let $P_{\tau'}$ be the projection to lowest $\widetilde{K}'$-type τ' in $L(\tau')$. View $L(\tau')$ as a $\widetilde{M}'$-submodule of $\mathscr{Y}^\infty$ by embedding into $\tau \otimes L(\tau') \subset \mathscr{Y}^\infty$ via tensor with a fixed vector in τ. Let $\{ v_j \}$ be a orthonormal basis of $\tau' \subset L(\tau')$. Let

$$\psi(g) = \mathrm{Tr}(P_{\tau'}\pi(g)P_{\tau'}) = \sum_{j=1}^{\dim \tau'} (\omega(g)v_j, v_j) \quad \forall g \in \widetilde{G}'.$$

We will show that the integration

$$\begin{aligned} \int_{\widetilde{G}'} \psi(g)\overline{\rho}(g)dg &= \sum_{j=1}^{\dim \tau'} \int_{\widetilde{G}'} (\omega(g)\overline{\rho}(g)v_j, v_j)dg \\ &= \sum_{j=1}^{\dim \tau'} \overline{\int_{\widetilde{G}'} (v_j, \omega(g)\overline{\rho}(g)v_j)dg} = \sum_{j=1}^{\dim \tau'} \overline{(v_j, v_j)_\rho}. \end{aligned} \tag{2.6}$$

is non-zero. Therefore there is some $v_j \notin \mathcal{R}_\rho$ and the image of v_j in $\mathscr{Y}^\infty/\mathcal{R}_\rho$ will generate a non-zero subspace with $\widetilde{K}$-type τ. This will finish the proof.

Now we are going to prove (2.6) is non-zero. Fix an Iwasawa decomposition

$$\widetilde{M'} = \widetilde{M}'^{(1,1)}AN,$$

where $\widetilde{M}'^{(1,1)}$ is a maximal compact subgroup of $\widetilde{M}$, A is split torus and N is a unipotent group. We may assume $A' = A \cap \widetilde{G}'$ is a split torus of $\widetilde{G}'$. Then by Lemma 3.3 [Li90],

$$\psi(k_1ak_2) = \int_{\widetilde{M}'^{(1,1)}} \left(\tau'(m^{-1}k_2k_1m)\xi, \xi\right) \phi(a,m)dm \quad \forall k_1, k_2 \in \widetilde{M}'^{(1,1)}, a \in A$$

where $\phi(a,m)$ is a positive (analytic) function on $A \times \widetilde{M}'^{(1,1)}$ and ξ is a unit lowest weight vector of τ'.

Let $\widetilde{G}' = \widetilde{K}'A'N'$ be an Iwasawa decomposition of $\widetilde{G}'$. Then $\widetilde{G}' = \widetilde{K}'A^+\widetilde{K}'$ where A^+ is a subset of $A' \subset A$. Now the Haar measure on $\widetilde{G}'$ can be written as $dx = \delta(a)dk_1\, da\, dk_2$ with the Jacobian δ a nonnegative function on A^+. Since ρ is a unitary character, $\rho(a) = 1$ for any $a \in A'$ and so,

$$\begin{aligned} &\int_{\widetilde{G}'} \psi(x)\overline{\rho}(x)\, dx \\ =& \int_{A^+} \int_{\widetilde{K}' \times \widetilde{K}'} \psi(k_1ak_2)\overline{\rho}(k_1ak_2)\delta(a)\, dk_1\, dk_2\, da \\ =& \int_{A^+} \int_{\widetilde{K}' \times \widetilde{K}'} \int_{\widetilde{M}'^{(1,1)}} (\tau'(m^{-1}k_2k_1m)\xi, \xi)\phi(a,m)\overline{\rho}(k_1k_2)\delta(a)\, dk_1\, dk_2\, da \\ =& \int_{A^+} \int_{\widetilde{M}'^{(1,1)}} \left(\int_{\widetilde{K}} (\tau'(m^{-1}km)\xi, \xi)\overline{\rho}(k)dk \right) \phi(a,m)\delta(a)\, dm\, da \end{aligned} \tag{2.7}$$

Let P_ρ be the projection map to ρ isotypic component in τ'. Then

$$\int_{\widetilde{K}} (\tau'(m^{-1}km)\xi, \xi)\overline{\rho}(k)\, dk = \|P_\rho \tau'(m)\xi\|^2$$

is nonnegative and not identically zero on $\widetilde{M}'^{(1,1)}$ since τ' is irreducible and ρ occur in τ'. Hence the integration (2.6) is nonzero, since the integrand is smooth nonnegative and not identically zero. □

2.3.7 Moment maps

In this section, we will define moments maps for dual pairs. We first define moment maps for compact dual pairs, and then define the moment map of non-compact dual pair by moment maps of compact dual pairs via diamond dual pair. In the end of this section, we will review the notion of theta lifting of nilpotent orbits and study some geometric properties of the corresponding double fibration.

2.3.7.1 Moment map for compact dual pairs and classical invariant theory

As discussed in Section 2.3.4, the space of $\widetilde{\mathrm{U}}$ finite vectors of the oscillator representation $\mathscr{Y}$ could be view as the ring of polynomials on a $\frac{1}{2}(\dim_{\mathbb{R}} W)$-dimensional complex vector space $W^{\mathbb{C}}$ corresponding to a fixing totally complex polarization of the real symplectic space W. Then double covers of compact groups K in reductive dual pairs act on $\mathscr{Y}$ linearly up to a twisting of genuine character. Here, we let the complexification $K_{\mathbb{C}}$ of K act on $\mathscr{Y} = \mathbb{C}[W^{\mathbb{C}}]$ linearly.

Now consider a compact dual pair (G, G') with G' compact. In this case, $K' = G'$ and G are all Hermitian symmetric. This means, there is a K-invariant decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

with $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Now $\mathfrak{p}^-$ act on $\mathscr{Y}$ by $K'_{\mathbb{C}}$-invariant degree 2 differential operators and $\mathfrak{p}^+$ act on $\mathscr{Y}$ by multiplying $K'_{\mathbb{C}}$-invariant degree 2 polynomials. Let

$$\mathcal{H} = \left\{\, f \in \mathbb{C}[W^{\mathbb{C}}] \;\middle|\; X \cdot f = 0, \forall X \in \mathfrak{p}^- \,\right\}$$

be the space of harmonics of K', which is all polynomials killed by $\mathfrak{p}^-$ and

$$\mathcal{I} = (\mathbb{C}[W^{\mathbb{C}}])^{K'_{\mathbb{C}}}$$

be the space of $K'_{\mathbb{C}}$-invariant polynomials.

The *moment maps* $\phi\colon W^{\mathbb{C}} \to (\mathfrak{p}^+)^*$ is a $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivalent map. The inverse image $\overline{\mathcal{N}} := \phi^{-1}(0)$ of $0 \in (\mathfrak{p}^+)^*$ is called the *null cone*. By identifying $W^{\mathbb{C}}$ and $(\mathfrak{p}^+)^*$ with certain vector spaces of matrix, we summarize the data in Table 2.4.

Case	$W^{\mathbb{C}}$	$(\mathfrak{p}^+)^*$	ϕ	action on $W^{\mathbb{C}}$	Stable
$\mathbb{R}$	$M_{n,m}$	Sym_n	$A \mapsto AA^T$	kAk'^{-1}	$m \geq 2n$
$\mathbb{C}$	$M_{n_1+n_2,m}$	M_{n_1,n_2}	$\begin{pmatrix} A \\ B \end{pmatrix} \mapsto AB^T$	$\begin{pmatrix} k_1Ak'^{-1} \\ k_2Bk'^T \end{pmatrix}$	$m \geq n_1 + n_2$
$\mathbb{H}$	$M_{n,2m}$	Alt_n	$A \mapsto AJA^T$	kAk'^{-1}	$m \geq n$

Table 2.4: Moment maps for compact dual pairs
Here $A \in W^{\mathbb{C}}$, $k \in K_{\mathbb{C}}$ and $k' \in K'_{\mathbb{C}}$ for Case $\mathbb{R}$ and $\mathbb{H}$;
$(A, B) \in M_{n_1,m} \times M_{n_2,m} = M_{n_1+n_2,m}$, $(k_1, k_2) \in \mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C})$, $k' \in \mathrm{GL}(m, \mathbb{C})$ for Case $\mathbb{C}$;
Sym_n denote the space of $n \times n$-symmetric matrix and Alt_n denote the space of $n \times n$-anti-symmetric matrix.

Lemma 13 ([How89a][How95]). *The moment map is induced by the morphism of $\mathbb{C}$-algebra*

$$\phi^*\colon \mathcal{S}(\mathfrak{p}^+) \to \mathbb{C}[W^{\mathbb{C}}]^{K'_{\mathbb{C}}}.$$

We have the following statements.

(a) ϕ^* is an isomorphism from $\mathfrak{p}^+$ to the $K'_{\mathbb{C}}$-invariant degree 2 polynomials in $\mathbb{C}[W^{\mathbb{C}}]$. Later we will identify $\mathfrak{p}^+$ as its image in $\mathbb{C}[W^{\mathbb{C}}]$.

(b) First Fundamental Theorem of classical invariant theory: ϕ^* is surjective. Hence $\phi\colon W \to W^{\mathbb{C}}/K'_{\mathbb{C}} \hookrightarrow (\mathfrak{p}^+)^*$ factor through the categorical quotient (affine-quotient) $W^{\mathbb{C}}/K'_{\mathbb{C}}$, which is a closed sub-variety of $(\mathfrak{p}^+)^*$.

(c) Let $\mathbb{C}[\overline{\mathcal{N}}]$ be the ring of regular functions on the null-cone $\overline{\mathcal{N}}$ and $i^*_{\overline{\mathcal{N}}}\colon \mathbb{C}[W^{\mathbb{C}}] \to \mathbb{C}[\overline{\mathcal{N}}]$ be the restriction map. By Kostant [Kos63],

$$\mathbb{C}[W^{\mathbb{C}}] = \mathcal{H} \oplus \mathfrak{p}^+\mathbb{C}[W^{\mathbb{C}}].$$

Therefore, $i^*_{\overline{\mathcal{N}}}|_{\mathcal{H}}$ is surjective.

(d) The $\mathbb{C}$-linear map $\mathcal{H} \otimes \mathcal{I} \to \mathbb{C}[W^{\mathbb{C}}]$ given by multiplication is always surjective.

(e) Under "Stable" condition listed in Table 2.4, the quotient map has following properties:

(i) $\mathcal{S}(\mathfrak{p}^+) \cong \mathcal{I}$ *and* ϕ *is a surjection onto* $(\mathfrak{p}^+)^*$.

(ii) The map $\mathcal{H} \otimes \mathcal{I} \to \mathbb{C}[W^{\mathbb{C}}]$ *is an isomorphism. Therefore,* $\phi\colon W^{\mathbb{C}} \to (\mathfrak{p}^+)^*$ *is a flat morphism.*

(iii) $i^*_{\mathcal{N}}\colon \mathcal{H} \to \mathbb{C}[\overline{\mathcal{N}}]$ *is an isomorphism of* $K_{\mathbb{C}} \times K'_{\mathbb{C}}$*-module.*

(iv) $K_{\mathbb{C}}$ *has an open dense orbit* $\mathcal{N}$ *in* $\overline{\mathcal{N}}$ *consists of all full rank matrices.*

2.3.7.2 Moment map for general dual pairs

The moment map for general dual pairs could be define via compact dual pairs.

We adapt the notation in [How89b]. Recall the *diamond dual pairs* in Figure 2.1. The pairs of groups similarly placed in the two diamonds are reductive dual pairs.

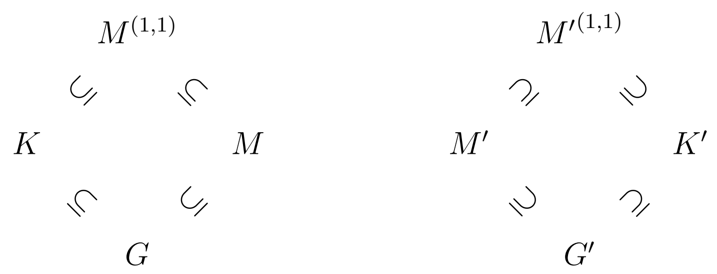


Figure 2.1: Diamond dual pairs

Note that (M, K') is a compact dual pair, denote the "$\mathfrak{p}^+$"(resp. $\mathfrak{p}^-$) part of $\mathfrak{m}$ to be $\mathfrak{m}^{(2,0)}$ (resp. $\mathfrak{m}^{(0,2)}$), therefore we have a moment map $\phi\colon W^{\mathbb{C}} \to \mathfrak{m}^{(2,0)}$. Fact 3 in Howe's paper[How89b] states that in $\mathfrak{sp}$, we have

$$\mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)} = \mathfrak{p} \oplus \mathfrak{m}^{(0,2)}. \tag{2.8}$$

The projection of $\mathfrak{p}$ into $\mathfrak{m}^{(2,0)}$ under the decomposition of the left hand side of (4.6) is a K-equivariant isomorphism. We will identify $\mathfrak{p}$ with $\mathfrak{m}^{(2,0)}$ via this projection. Therefore, we get the moment map for G:

$$\phi\colon W^{\mathbb{C}} \to (\mathfrak{m}^{(2,0)})^* \cong \mathfrak{p}^*.$$

We define the moment map $\phi'\colon W^{\mathbb{C}} \to \mathfrak{p}'^*$ for G' similarly via pair (K, M'). By identifying $W^{\mathbb{C}}$, $\mathfrak{p}^*$ and $\mathfrak{p}'^*$ with space of matrices, we list the explicit formula in Table 2.5 for some non-compact dual pairs[8].

G	G'	$W^{\mathbb{C}}$ $w \in W$	$\mathfrak{p}^*$ $\phi(w)$	$\mathfrak{p}'^*$ $\phi'(w)$
$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{O}(p, q)$	$M_{p,n} \times M_{q,n}$ (A, B)	$\mathrm{Sym}_n \times \mathrm{Sym}_n$ $(A^T A, B^T B)$	$M_{p,q}$ AB^T
$\mathrm{U}(n_1, n_2)$	$\mathrm{U}(p, q)$	$M_{p,n_1} \times M_{p,n_2} \times M_{q,n_1} \times M_{q,n_2}$ (A, B, C, D)	$M_{n_1,n_2} \times M_{n_2,n_1}$ $(A^T B, D^T C)$	$M_{p,q} \times M_{q,p}$ (AC^T, DB^T)
$\mathrm{O}^*(2n)$	$\mathrm{Sp}(p, q)$	$M_{2p,n} \times M_{2q,n}$ (A, B)	$\mathrm{Alt}_n \times \mathrm{Alt}_n$ $(A^T J_{2p} A, B^T J_{2q} B)$	$M_{2p,2q}$ AB^T

Table 2.5: Moment maps for non-compact dual pairs

2.3.7.3 Theta lifting of nilpotent orbits

In this section, we will discuses the notion of theta lifting of nilpotent orbits, this notion is studied by many authors and usually appeared as "resolution of singularity", some related papers includes [NOZ06] [DKP05][Oht91]. We retain the notation in Section 2.3.7.2. To simplify the notation, we will identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ by trace form, and so, identify $\mathfrak{p}^*$ with $\mathfrak{p}$ in this section.

First recall the definition of "nilpotent". Let a reductive algebraic group G act linearly on a vector space V, then a element in V is called a *nilpotent element* if the closure of its G-orbit contains zero and the orbit is called an *nilpotent orbit*. The union of all nilpotent orbit is called *null-cone*, denote by $\mathcal{N}_V$. Also let $\mathfrak{N}_G(V)$ be the set of nilpotent orbits in V with respect to the G action.

Since the reductive group $K_{\mathbb{C}}$ (resp. $K'_{\mathbb{C}}$) act on $\mathfrak{p}$ (resp. $\mathfrak{p}'$) linearly, we have null-cone $\mathcal{N}_{\mathfrak{p}}$ (resp. $\mathcal{N}_{\mathfrak{p}'}$) and set of nilpotent orbites $\mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p})$ (resp. $\mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}')$). For a non-compact dual pair (G, G'), moment maps ϕ, ϕ' gives following double fibration:

$$\mathfrak{p} \xleftarrow{\ \phi\ } W^{\mathbb{C}} \xrightarrow{\ \phi'\ } \mathfrak{p}'$$

For every nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}'$ in $\mathfrak{p}'$, $\phi(\phi'^{-1}(\overline{\mathcal{O}'}))$ is a (non-empty Zariski) closed $K_{\mathbb{C}}$-invariant subset of $\mathfrak{p}$.[9] When $\phi(\phi'^{-1}(\overline{\mathcal{O}'}))$ is the clo-

[8] In Table 2.5 $J_2p = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}$

[9] Obviously, 0 is in the set, i.e. it is non-empty. The claim of Zariski closeness is from the fact that ϕ is factor through the affine quotient of $K'_{\mathbb{C}}$ and the image of the affine quotient is an closed subset of $\mathfrak{p}$ by classical invariant theory

sure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}$, it is nature to define $\mathcal{O}$ to be the theta lifts of $\mathcal{O}'$. While, in general, $\phi(\phi'^{-1}(\overline{\mathcal{O}'}))$ may have several (finite many) open orbits, then it is not clear how to define the notion of theta lift for such nilpotent orbit properly. Fortunately, for a $K'_{\mathbb{C}}$-orbit $\mathcal{O}'$, $\phi(\phi'^{-1}(\overline{\mathcal{O}'}))$ is always the closure of a single $K_{\mathbb{C}}$-orbit when (G, G') is a non-compact real reductive dual pair in stable range with G' the smaller member.

We recode above discussion in the following definition.

Definition 14 (c.f. [Oht91] [DKP05][NOZ06])**.** For any nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}'$ in $\mathfrak{p}'$, a nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}$ is called the *theta lift of nilpotent orbit* $\mathcal{O}'$ if $\phi(\phi'^{-1}(\overline{\mathcal{O}'}))$ equal to the closure of $\mathcal{O}$.

When (G, G') is a non-compact real reductive dual pair in stable range with G' the smaller member, we have an injective map

$$\theta\colon \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}') \to \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p})$$

defined by $\mathcal{O}' \mapsto \mathcal{O}$.

Moreover, we extend θ linearly to the spaces of cycles (formal sums) of nilpotent orbits, also denote it by θ.

Remark: One can define the notion of theta lifting for nilpotent G'-orbit in ${\mathfrak{g}'_0}^*$ to nilpotent G-orbit in $\mathfrak{g}_0^*$ in a similarly way. Daszkiewicz, Kraśkiewicz and Przebinda [DKP05], showed that the two notion of theta lifts of nilpotent orbits are compatible under Kostant-Sekiguchi correspondence in stable range. Furthermore, they gives examples to show that the relationship could be tricky outside the stable range.

In the rest of this section, we will study the structure of isotropic groups (stabilizers) for Type I dual pairs in stable range. One may obtain the results from the classification of unipotent orbits and the explicit construction of theta lifts of orbits[Oht91]. But, we will prove these results from properties of null-cone and classical invariant theory. I learnt this conceptually simpler method from [Nis07].

Firstly, we review the constructions of moment map in [DKP05]. For a $\mathbb{Z}/4\mathbb{Z}$-graded

$$U = U_0 \oplus U_1 \oplus U_2 \oplus U_3, \tag{2.9}$$

let

$$\mathrm{End}(U)_a = \{\, X \in \mathrm{End}(U) \mid X(U_b) \subset U_{a+b}\ \forall\, b \,\}.$$

If $\langle\ ,\ \rangle$ is a sesqui-linear form on U. Define $S \in \mathrm{End}(V)_0$ by $Sv = (-1)^a v$

for all $v \in U_a$. Define

$$\begin{aligned} \mathfrak{g}(U)_a &= \{\, X \in \mathrm{End}(U)_a \mid \langle Xu, v\rangle + \langle S^a u, Xv\rangle = 0, \forall u, v \in U \,\}, \\ G(U)_0 &= \{\, g \in \mathrm{End}(U)_0 \cap \mathrm{GL}(U) \mid \langle gu, gv\rangle = \langle u, v\rangle, \forall u, v \in U \,\}. \end{aligned}$$

For dual pairs in Table 2.5, define U to be a $\mathbb{Z}/4\mathbb{Z}$ graded space as in Table 2.6. Associate an form $\langle,\rangle$ on U. For Case $\mathbb{C}$, the form $\langle\ ,\ \rangle$ is zero. For Case $\mathbb{R}$ (resp. $\mathbb{H}$), $\langle\ ,\ \rangle$ is non-degenerate and symmetric (resp. skew-symmetric) on U_0, U_2; $\langle\ ,\ \rangle$ is non-degenerate skew-symmetric (resp. symmetric) on $U_1 \oplus U_3$ and U_1, U_3 are isotropic subspace.

	U_0	U_1	U_2	U_3
Case $\mathbb{R}$	$\mathbb{C}^p$	$\mathbb{C}^n$	$\mathbb{C}^q$	$\mathbb{C}^n$
Case $\mathbb{C}$	$\mathbb{C}^p$	$\mathbb{C}^{n_1}$	$\mathbb{C}^q$	$\mathbb{C}^{n_2}$
Case $\mathbb{H}$	$\mathbb{C}^{2p}$	$\mathbb{C}^n$	$\mathbb{C}^{2q}$	$\mathbb{C}^n$

Table 2.6: $\mathbb{Z}/4\mathbb{Z}$ graded vector space for Type I dual pairs.

Let $U_{\text{even}} = U_0 \oplus U_2$ and $U_{\text{odd}} = U_1 \oplus U_3$. Then

$$W^{\mathbb{C}} \cong \mathfrak{g}(U)_1, \quad \mathfrak{p} \oplus \mathfrak{p}' \cong \mathfrak{g}(U)_2 \cong \mathfrak{g}(U_{\text{even}})_2 \oplus \mathfrak{g}(U_{\text{odd}})_2, \quad K_{\mathbb{C}} \times K'_{\mathbb{C}} \cong G(U)_0.$$

Moreover, under above identification, $x \mapsto x^2$ gives the *moment map* (where $\triangle\colon W^{\mathbb{C}} \to W^{\mathbb{C}} \times W^{\mathbb{C}}$ is the diagonal map)

$$\phi \times \phi' \circ \triangle\colon W^{\mathbb{C}} \cong \mathfrak{g}(U)_1 \to \mathfrak{g}(U)_2 \cong \mathfrak{p} \oplus \mathfrak{p}'.$$

For a type I dual pairs (G, G') we say it is in stable range with G' the smaller member if it satisfies conditions in Table 2.7. The conditions is equivalent to say the reductive dual pair (K, M') in the diamond dual pairs(Figure 2.1), form an Stable compact dual pair(see Table 2.4).

	G	G'	Stable range	V	V'
Case $\mathbb{R}$	$\mathrm{O}(p,q)$	$\mathrm{Sp}(2n,\mathbb{R})$	$p, q \geq 2n$	U_{even}	U_{odd}
	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{O}(p,q)$	$n \geq p+q$	U_{odd}	U_{even}
Case $\mathbb{C}$	$\mathrm{U}(p,q)$	$\mathrm{U}(n_1,n_2)$	$p, q \geq n_1+n_2$	U_{even}	U_{odd}
Case $\mathbb{H}$	$\mathrm{Sp}(p,q)$	$\mathrm{O}^*(2n)$	$p, q \geq n$	U_{even}	U_{odd}
	$\mathrm{O}^*(2n)$	$\mathrm{Sp}(p,q)$	$n \geq 2(p+q)$	U_{odd}	U_{even}

Table 2.7: Stable range for Type I dual pairs

Therefore, we have following facts.

Lemma 15. *For real reductive dual pairs (G, G') and V, V' in Table 2.7.*

(i) The map $\phi'\colon W^{\mathbb{C}} \to \mathfrak{p}'$ is a flat morphism.

(ii) The null-cone $\phi'^{-1}(0)$ has an dense $K_{\mathbb{C}}$-orbit $\mathcal{N}$.

(iii) For a element $X \in \mathcal{N}$, X has full rank as element $Hom_{\mathbb{C}}(V, V') \oplus \operatorname{Hom}_{\mathbb{C}}(V', V)$. This means $X|_V$ (equivalently, $X|'_V$) is surjection (injection) for case $\mathbb{R}$ and $\mathbb{H}$; $X|_V$ is surjection and $X|_{V'}$ injection for $\mathbb{C}$. □

Lemma 16. *Let $\mathcal{O}' \subset \mathfrak{p}'$ be an $K'_{\mathbb{C}}$ nilpotent orbit in $\mathfrak{p}'$. Then $\mathcal{O}'$ admit theta lift.*

Moreover, let $x' \in \mathcal{O}' \in \mathfrak{p}'$.

(i) $\phi'^{-1}(x')$ has an open dense $K_{\mathbb{C}}$-orbit $\mathcal{X}$;

(ii) let $X \in \mathcal{X}$, $X|_V \in \operatorname{Hom}_{\mathbb{C}}(V, V')$ is a surjection;

(iii) let $x = \phi(X)$, then x generate nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{p}$ such that $\overline{\mathcal{O}} = \phi(\phi'^{-1}(\overline{\mathcal{O}'}))$.

Proof. All the claims could be checked by the explicit description of nilpotent orbits and moment map (c.f. [DKP05, Table 4]). Part (i) and (iii) also a could be proved by deformation method (see [Nis07, Theorem 2.4], except for p or $q = 2n$ in case $\mathbb{R}$). For part (ii), since having full rank is an open condition, i.e. there is an open subset $\mathcal{B}$, such that $\mathcal{B} \cap \mathcal{N} \neq \emptyset$ and X has full rank for any $X \in \mathcal{B}$. On the other hand, note that ϕ' is a flat morphism (so is open map); and $\mathcal{D} := K'_{\mathbb{C}}\mathcal{X}$ is dense in $\overline{\phi'^{-1}(\mathcal{O})}$. So $\overline{\mathcal{D}} = \phi'^{-1}(\overline{\mathcal{O}'})$. Since $\mathcal{O}'$ is nilpotent, $\mathcal{N} \subset \phi'^{-1}(0) \subset \overline{\mathcal{D}}$. Therefore $\mathcal{B} \cap \mathcal{D} \neq \emptyset$. Since $\mathcal{D}$ is an orbit, all points X in $\mathcal{D}$ satisfies X has full rank. This finished the proof. □

Now fix $X \in \mathcal{X}$, and let $x' = \phi'(X) \in \mathcal{O}'$ and $x = \phi(X)$ as in Lemma 16. Let

$$K_x = \operatorname{Stab}_{K_{\mathbb{C}}}(x) \qquad K'_{x'} = \operatorname{Stab}_{K'_{\mathbb{C}}}(x') \qquad S_X = \operatorname{Stab}_{K_{\mathbb{C}} \times K'_{\mathbb{C}}}(X).$$

We define a group homomorphism

$$\alpha \colon K_x \to K'_{x'} \quad \text{such that} \quad \alpha(k)Xv = Xkv, \quad \forall v \in V, k \in K_x. \tag{2.10}$$

Let $k' = \alpha(k)$, it is routine to check k' is well defined. First notice that k' as an linear map on V' is unique if it exist, since $X|_V$ is surjective. For Case $\mathbb{R}$ and $\mathbb{H}$, we have non-degenerate form on V and V'.

(a) k' is well defined, i.e. for any $v \in \operatorname{Ker} X$, $Xkv = 0$: We only need to show $\langle Xkv, Xv_2 \rangle = 0$. Since k stabilize $x = XX|_V$, $\langle Xkv, Xv_2 \rangle =$

$(-1)^s \langle XXkv, v_2\rangle = (-1)^s \langle kXXv, v_2\rangle = 0$ where $s \in \mathbb{Z}/2\mathbb{Z}$ depend on the parity of V.

(b) To show $k' \in G(V)_0$, we only need to show $\langle k'Xv_1, k'Xv_2\rangle = \langle Xv_1, Xv_2\rangle$. In fact, $\langle k'Xv_1, k'Xv_2\rangle = (-1)^s \langle XXkv_1, kv_2\rangle = (-1)^s \langle kXXv_1, kv_2\rangle = \langle XXv_1, v_2\rangle = \langle Xv_1, Xv_2\rangle$.

(c) $k' \in K'_{x'}$. We only need to show, for all $v'_1 = Xv_1, v'_2 \in V'$, we have $\langle k'X^2v'_1, v'_2\rangle = \langle X^2k'v'_1, v'_2\rangle$. In fact, $\langle k'X^2v'_1, v'_2\rangle = \langle XkXv'_1, v'_2\rangle = \langle XkX^2v_1, v'_2\rangle = \langle X^3kv_1, v'_2\rangle = \langle X^2k'Xv_1, v'_2\rangle = \langle X^2k'v'_1, v'_2\rangle$.

For Case $\mathbb{C}$,

(a) k' is well defined. For any $v \in V$ such that $Xv = 0$, $X|_{V'}(Xkv) = kX^2v = 0$. So $Xkv = 0$, since $X|_{V'}$ is injective.

(b) k' is in $K'_{\mathbb{C}}$, since it is invertible by the definition.

(c) $k' \in K_{x'}$. In fact, for any $v' = Xv \in V'$, $k'X^2v' = XkX^2v = X^3kv = X^2k'Xv = X^2k'v'$.

Let $K_x \times_\alpha K'_{x'} = \{\ (k, k') \mid \alpha(k) = k'\ \} \subset K_x \times K'_{x'}$ be following pull back.

$$\begin{array}{ccc} K_x \times_\alpha K'_{x'} & \longrightarrow & K_x \\ \downarrow & & \downarrow \alpha \\ K'_{x'} & = & K'_{x'} \end{array}$$

Lemma 17. *We have following equation describe the structure of S_X, with α defined in (2.10):*

$$S_X = K_x \times_\alpha K'_{x'} \subset K_x \times K'_{x'}.$$

Proof. Let $(k, k') \in S_X$, i.e $k'Xk^{-1} = X$. By the definition of α, $k'Xk = k'\alpha(k^{-1})X$ and X is surjective. Hence $k' = \alpha(k)$, i.e. $S_W \subset K_x \times_\alpha K'_x$. On the other hand, $K_x \times_\alpha K'_{x'} \subset S_W$ is also clear by the definition of α. This finished the proof. □

2.4 Basic facts about derived functors

In this section, we review basic facts about the derived functors on the category of $(\mathfrak{g}, K)$-modules. We follow Vogan[Vog81] and Borel-Wallach[BW00].

2.4.1 Zuckerman functor

For $V \in \mathscr{C}(\mathfrak{g}, K)$, let K^0 be the identity component of K. Define

$$C^q(\mathfrak{g}, \mathfrak{k}; V) := \mathrm{Hom}_{\mathfrak{k}}(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), V),$$
$$C^q(\mathfrak{g}, K; V) := \mathrm{Hom}_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), V) = C^q(\mathfrak{g}, \mathfrak{k}; V)^{K/K^0}.$$

The *relative Lie algebra cohomology* $H^q(\mathfrak{g}, \mathfrak{k}; V)$(resp. $H^q(\mathfrak{g}, K; V)$) is the cohomology of above chain complex $C^q(\mathfrak{g}, \mathfrak{k}; V)$ (resp. $C^q(\mathfrak{p}, K; V)$) with the chain map induced from the Koszul complex of $\mathfrak{g}$. Clearly

$$H^q(\mathfrak{g}, K; V) = H^q(\mathfrak{g}, \mathfrak{k}; V)^{K/K^0} \tag{2.11}$$

since taking K/K^0 invariant is a exact functor.

Let M be a subgroup of K. *Zuckerman functor* $\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K} \colon \mathscr{C}(\mathfrak{g}, M) \to \mathscr{C}(\mathfrak{g}, K)$ is the right adjoint functor of the forgetful functor $F_{\mathfrak{g},K}^{\mathfrak{g},M} : \mathscr{C}(\mathfrak{g}, K) \to \mathscr{C}(\mathfrak{g}, M)$. This functor is only left exact and usually will be zero on modules in $\mathscr{C}(\mathfrak{g}, M)$. Therefore, we consider the right derived functors of Γ and they can construct interesting objects in $\mathscr{C}(\mathfrak{g}, K)$. Let $R^q\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}$ be the q-th right derived functor of $\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}$. They can be realized by relative Lie algebra cohomology.

First note that, a $(\mathfrak{g}, M)$-module can be view as $(\mathfrak{k}, M)$-module via forgetful functor $F_{\mathfrak{g},M}^{\mathfrak{k},M}$. As $(\mathfrak{k}, K)$-module, there is a canonical isomorphism:

$$R^q\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}V \cong R^q\Gamma_{\mathfrak{k},M}^{\mathfrak{k},K}V.$$

Let $\mathcal{H}(K) = \bigoplus_{\gamma \in \widehat{K}} V_\gamma^* \otimes V_\gamma$ be the Hecke algebra of K consists of all K-finite functions on K under left and right translation. Now, for any $(\pi, V) \in \mathscr{C}(\mathfrak{g}, M)$,

$$\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}V = \big((V \otimes \mathcal{H}(K))^{\mathfrak{k}}\big)^M = \bigoplus_{\gamma \in \widehat{K}} \mathrm{Hom}_{\mathfrak{k},M}(V_\gamma, V) \otimes V_\gamma, \tag{2.12}$$

$$R^q\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}V = H^q(\mathfrak{k}, M; V \otimes \mathcal{H}(K)) = \bigoplus_{\gamma \in \widehat{K}} H^q(\mathfrak{k}, M; V \otimes V_\gamma^*) \otimes V_\gamma, \tag{2.13}$$

where $\mathcal{H}(K)$ is view as an $(\mathfrak{k}, M)$-module via left regular action. Above map is K-module morphism with K act by left translation on $\mathcal{H}(K)$, the $\mathfrak{g}$ structure on V could be defined functorially.

If K is connected, $\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}V$ could be view as the subspace of all $v \in V$ such that the action of $\mathfrak{k}$ can be globalize to K. Now $\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}V$ has $\mathfrak{g}$-structure inherited from V, since this subspace is $\mathfrak{g}$-invariant. While K is not always connected, the calculation could be reduced to the connected case. By Frobenius reciprocity,

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{k},M}(V_\gamma, V) =&\mathrm{Hom}_{\mathfrak{k},M}(V_\gamma, \bigoplus_{V_i\in\widehat{MK^0}} \mathrm{Hom}_{\mathfrak{k}}(V_i, V)\otimes V_i)\\ =&\mathrm{Hom}_{\mathfrak{k},K}(V_\gamma, \mathrm{Ind}_{MK^0}^{K}\Gamma_{\mathfrak{k},M}^{\mathfrak{g},MK^0}V).\end{aligned}$$

Let $M_1 = K^0\cap M$. Since M/M_1 is a finite group, $\Gamma_{\mathfrak{k},M_1}^{\mathfrak{k},K^0}V\subset V$ is M-invariant with M-action inherited from V and $\Gamma_{\mathfrak{g},M}^{\mathfrak{g},MK^0}V\cong\Gamma_{\mathfrak{g},M_1}^{\mathfrak{g},K^0}V$. Above calculation implies

$$\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}V = \mathrm{Ind}_{MK^0}^{K}\Gamma_{\mathfrak{g},M_1}^{\mathfrak{g},K^0}(V).$$

Since $\mathrm{Ind}_{MK^0}^{K}$ is exact, we get

$$R^q\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}V = \mathrm{Ind}_{MK^0}^{K}\left(R^q\Gamma_{\mathfrak{g},M_1}^{\mathfrak{g},K^0}(V)\right).$$

Combining above equation with (2.11) and (2.13), the computation of the K-spectrum of $(R^q\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K})V$ is reduced to calculate cohomology $H^q(\mathfrak{k},\mathfrak{m}; V\otimes V_\gamma^*)$ for $\gamma\in\widehat{K^0}$. In fact, M will always intersect all connected components of K in our application. So we will assume from now on this is true. In this case,

$$R^q\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}V = R^q\Gamma_{\mathfrak{g},M_1}^{\mathfrak{g},K^0}(V).$$

When $\mathfrak{g}, K$ and M is clear in the context, we will denote $\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}$ and $\Gamma_{\mathfrak{k},M}^{\mathfrak{k},K}$ by Γ; denote their right derived functors by Γ^q.

2.4.2 A decomposition of derived functor module

The derived functor module $\Gamma^q V$ is not irreducible in general. In this section, we will describe a direct sum decomposition of the derived functor module under following assumptions (c.f. [WZ04]):

(**A**1) There is a real reductive group K_1 such that $\mathfrak{k}_1 = Lie(K_1)$ is a real form of $\mathfrak{k}$. Moreover, M is a maximal compact subgroup of K_1, i.e. (K_1, M) form a symmetric pair of non-compact type;

(**A**2) $(\pi, V)\in\mathscr{C}(\mathfrak{g}, M)$ can be decompose into a direct sum of irreducible unitarizable $(\mathfrak{k}, M)$-modules, say

$$V = \bigoplus_j V_j,$$

such that each V_j is the Harish-Chandra module of an irreducible unitary K_1-representation.

For $(\pi, V) \in \mathscr{C}(\mathfrak{g}, M)$, let $\mathcal{H}(K, V) := V \otimes \mathcal{H}(K)$ be the space of K-finite function on K with value in V. The $(\mathfrak{k}, M)$-module structure on it is given by π tensor with left translation. Define a $\mathfrak{g}$ structure μ on $\mathcal{H}(K, V)$ as following

$$(\mu(X)f)(k) = \pi(\mathrm{Ad}_k X)f(k) \quad \forall k \in K, X \in \mathfrak{g}, f \in \mathcal{H}(K, V).$$

One can check that μ is a Lie algebra action and commutes with the $(\mathfrak{k}, M)$-action. Moreover, the induced $\mathfrak{g}$-action on $\Gamma^q V$ is compatible with the induced K-action from right translations on $\mathcal{H}(K, V)$. This construction gives the $(\mathfrak{g}, K)$-module structure on $\Gamma^q V$.

By Wigner' Lemma (c.f. [BW00, Theorem 5.3]), $H^q(\mathfrak{k}, M; V_j \otimes V_\gamma^*)$ vanish if V_j and V_γ have different infinitesimal characters (or central characters). So we decompose $\mathcal{H}(K, V)$ into direct sum of $\Omega(V)$ and $\Omega'(V)$ with

$$\Omega(V) = \bigoplus_{\gamma \in \widehat{K}} \bigoplus_{V_\gamma \in \widehat{K}_{V_j}} V_j \otimes V_\gamma^* \otimes V_\gamma \tag{2.14}$$

$$\Omega'(V) = \bigoplus_{\gamma \in \widehat{K}} \bigoplus_{V_\gamma \notin \widehat{K}_{W_j}} V_j \otimes V_\gamma^* \otimes V_\gamma \tag{2.15}$$

where $\widehat{K}_{V_j}$ is the set of irreducible K-modules having the same infinitesimal character and central character as V_j. Then

$$\Gamma^q V = H^q(\mathfrak{k}, M; V \otimes \mathcal{H}(K)) = H^q(\mathfrak{k}, M; \Omega(V)).$$

Now recall following theorem on the cohomology of unitarizable $(\mathfrak{g}, K)$-module. Fix a G-invariant and Cartan involution invariant non-degenerate symmetric bilinear form $B(\cdot,\cdot)$ on $\mathfrak{g}$ such that restrict on $\mathfrak{k}$(resp. $\mathfrak{p}$) is negative (resp. positive) definite. Let $C = \sum y_s y_s'$ be the *Casimir element* in $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$, where $\{ y_s \}$ is a basis of $\mathfrak{g}$ and $\{ y_s' \}$ is its dual basis with respect to $B(\cdot,\cdot)$.

Theorem 18 (Proposition II.3.1[BW00])**.** *Let (π, V) be a unitarizable $(\mathfrak{g}, \mathfrak{k})$-module and (ρ, E) a finite dimensional G-module. Let C be the Casimir element. Assume that $\pi(C) = s \cdot \mathrm{Id}, \rho(C) = r \cdot \mathrm{Id}$.*

(a) If $r \neq s$, then $H^q(\mathfrak{g}, \mathfrak{k}; V \otimes E) = 0$ for all q's.

(b) If $r = s$, then all co-chains are closed, harmonic, and we have

$$H^q(\mathfrak{g}, \mathfrak{k}; V \otimes E) = \mathrm{Hom}_{\mathfrak{k}}(\bigwedge^q \mathfrak{p}, V \otimes E) \quad \forall q \in \mathbb{N}.$$

For $(\mathfrak{k}, M)$-module V_j and finite dimensional K_1-module[10] V_γ, the condition $V_\gamma \in \widehat{K}_{V_j}$ implies C has same the scaler action on them. Therefore

$$\Gamma^q V = \left(\mathrm{Hom}_{\mathfrak{m}}(\bigwedge^q \mathfrak{k}/\mathfrak{m}, \Omega(V))\right)^{M/M^0} = \mathrm{Hom}_M(\bigwedge^q \mathfrak{k}/\mathfrak{m}, \Omega(V)).$$

There is an $(\mathfrak{g}, K)$-module structure on $\Omega(V)$ by the projection of μ. The $\mathfrak{g}$-module structure on $\Gamma^q V$ is given by post composition.

Definition 19. Now for any M-submodule $W \in \bigwedge^q(\mathfrak{k}/\mathfrak{m})$, define

$$\Gamma_W(V) = \mathrm{Hom}_M(W, \Omega(V)), \tag{2.16}$$

which is a $(\mathfrak{g}, K)$-submodule of $\Gamma^q V$ since M-action commute with $\mathfrak{g}$ and K actions on $\Omega(V)$.

Now decompose $\bigwedge^q \mathfrak{k}/\mathfrak{m} = \bigoplus W_i$ as direct sum of irreducible M-modules, we have following decomposition of $\Gamma^q V$ into direct sum of $(\mathfrak{g}, K)$-modules:

$$\Gamma^q V = \bigoplus \Gamma_{W_i} V. \tag{2.17}$$

2.4.3 $A_{\mathfrak{q}}(\lambda)$ and Vogan-Zuckerman's Theorem

Vogan and Zuckerman have classified irreducible unitary representations with non-zero cohomology in [VZ84] in terms of $A_{\mathfrak{q}}(\lambda)$. In this section, we will review their main theorems and compute the cohomology of a family of representations which we will use later.

Temporarily, let G be a real connected semisimple Lie group and θ be a Cartan involution on $\mathfrak{g}$. Fix a θ-stable fundamental Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, such that $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ is the Cartan subalgebra of $\mathfrak{k}$. Fixing a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ compatible with $\mathfrak{h}$. Fix a θ-stable system of positive roots compatible with $\mathfrak{q}$ (such that all roots in $\mathfrak{q}$ is non-negative). Denote the half sum of roots in a $\mathfrak{h}$-invariant subspace $\mathfrak{n} \subset \mathfrak{g}$ by $\rho(\mathfrak{n})$ and the half sum of all positive roots by ρ.

Definition 20 ([VZ84, Section 5]). A character $\lambda \colon \mathfrak{l} \to \mathbb{C}$ is called *admissible* if

[10] holomorphically extend K-module to $K_{\mathbb{C}}$-module and then restriction to K_1.

(a) λ is the differential of a unitary character of the analytic subgroup L of $\mathfrak{l}$;

(b) $\langle \alpha, \lambda|_{\mathfrak{t}} \rangle \geq 0$ for all $\alpha \in \Delta(\mathfrak{u})$, where $\langle , \rangle$ is the pairing on the restricted root system of $\mathfrak{t}$.

Let $A_{\mathfrak{q}}(\lambda) = \mathscr{R}_{\mathfrak{q}}^S(\lambda)$ be the cohomologically induced module, where $S = \dim \mathfrak{u} \cap \mathfrak{k}$, c.f. [Vog81].

Let

$$\mu(\mathfrak{q}, \lambda) = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p}) \tag{2.18}$$

and identify it with the $\mathfrak{k}$-module of highest weight $\mu(\mathfrak{q}, \lambda)$. Such $A_{\mathfrak{q}}(\lambda)$ is unique in the following sense:

Theorem 21 (Vogan-Zuckerman [VZ84] Theorem 5.3)**.** *$A_{\mathfrak{q}}(\lambda)$ is the unique irreducible $\mathfrak{g}$-module with the following properties:*

(a) The restriction of $A_{\mathfrak{q}}(\lambda)$ to $\mathfrak{k}$ contains $\mu(\mathfrak{q}, \lambda)$;

(b) $\mathcal{Z}(\mathfrak{g})$ acts by $\chi_{\lambda+\rho}$ in $A_{\mathfrak{q}}(\lambda)$

(c) If the representation of $\mathfrak{k}$ of highest weight σ occurs in $A_{\mathfrak{q}}(\lambda)$ restricted to $\mathfrak{k}$, then

$$\sigma = \lambda|_{\mathfrak{t}} + 2\rho(\mu \cap \mathfrak{p}) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_\beta \beta,$$

with n_β are non-negative integers.

There is a simple criterion to detect $A_{\mathfrak{q}}(\lambda)$ among unitary representations.

Theorem 22 ([VZ84] Proposition 6.1)**.** *Assume that λ is an admissible character of $\mathfrak{l}$ and zero on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$. Let X be an irreducible unitary $(\mathfrak{g}, K)$-module. Then $X \cong A_{\mathfrak{q}}(\lambda)$ if*

(a) The $\mathfrak{k}$-type $\mu(\mathfrak{q}, \lambda)$ occurs in X.

(b) X has infinitesimal character $\lambda + \rho$.

The cohomology of $A_{\mathfrak{q}}(\lambda)$ is computed as following.

Theorem 23 (Vogan-Zuckerman [VZ84])**.** *Let $R = \dim \mathfrak{u} \cap \mathfrak{p}$ and F be a finite dimensional irreducible represntation of $\mathfrak{g}$ with highest weight γ. Then*

$$H^q(\mathfrak{g}, \mathfrak{k}; A_{\mathfrak{q}}(\lambda) \otimes F^*) \cong H^{j-R}(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{t}, \mathbb{C}) \cong \mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{t}}(\bigwedge^{j-R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C})$$

if $\gamma = \lambda|_{\mathfrak{h}}$*; and*

$$H^q(\mathfrak{g}, \mathfrak{k}; A_{\mathfrak{q}}(\lambda) \otimes F^*) = 0,$$

otherwise.

All irreducible unitary representations with non-zero cohomology are certain $A_{\mathfrak{q}}(\lambda)$ by following theorem.

Theorem 24 ([VZ84, Theorem 5.6])**.** *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module, and F an irreducible finite dimensional representation of G. Suppose $H^*(\mathfrak{g}, K; X \otimes F) \neq 0$. Then there is a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$, such that*

(a) $F/\mathfrak{u}F$ is a one dimensional unitary representation of L; let $-\lambda\colon \mathfrak{l} \to \mathbb{C}$ for its differential.

(b) $X \cong A_{\mathfrak{q}}(\lambda)$.

In the rest of this section, we will apply above theorems to compute the cohomology of the Harish-Chandra modules of unitary Lowest weight modules of $\widetilde{U}(r, s)$. The result will be used in section 3.5.1.

Let p, q, r, s be positive integers such that $p \leq r$ and $q \leq s$. Let

$$\mu = (a_1, \cdots, a_p, -b_q, \cdots, -b_1)$$

such that $a_1 \geq \cdots \geq a_p \geq 0$ and $b_1 \geq \cdots \geq b_q \geq 0$. Let $L_{\widetilde{\mathrm{U}}(r,s)}(\mu)$ be the lowest weight $(\mathfrak{u}(r{+}s, \mathbb{C}), \widetilde{\mathrm{U}}(r){\times}\widetilde{\mathrm{U}}(s))$-module with lowest $\widetilde{\mathrm{U}}(r){\times}\widetilde{\mathrm{U}}(s)$-type

$$\tau := \tau_{\widetilde{\mathrm{U}}(r)}^{(a_1, \cdots, a_p, 0, \cdots, 0) + \frac{m}{2}} \otimes \tau_{\widetilde{\mathrm{U}}(s)}^{(0, \cdots, 0, -b_q, \cdots, -b_1) - \frac{m}{2}}.$$

In fact, $L_{\widetilde{\mathrm{U}}(r,s)}(\mu)$ is the theta lift of $\widetilde{\mathrm{U}}(m)$-module $\tau_{\widetilde{\mathrm{U}}(m)}^{\mu + \frac{r-s}{2}}$ via compact dual pair $(\mathrm{U}(r, s), \mathrm{U}(m))$ (See Section 2.3.5) and so, it is unitarizable. Let

$$\begin{aligned} a'_i =& a_i - (s - q) \qquad b'_i = b_i - (r - p) \\ \lambda =& (a'_1, \cdots, a'_q, 0, \cdots, 0, -b'_q, \cdots, -b'_1) + \frac{p - q}{2}. \end{aligned}$$

Temporally let $\mathfrak{g} = \mathfrak{gl}(r + s, \mathbb{C})$, $\mathfrak{k} = \mathfrak{gl}(r, \mathbb{C}) \oplus \mathfrak{gl}(s, \mathbb{C})$. Note that $\mathfrak{g}, \mathfrak{k}$ here will be $\mathfrak{k}, \mathfrak{m}$ in previous sections.

Lemma 25. *Let* $\Gamma := \Gamma^{\mathfrak{gl}(r+s, \mathbb{C}), \widetilde{U}(r+s)}_{\mathfrak{gl}(r+s, \mathbb{C}), \widetilde{U}(r) \times \widetilde{U}(s)}$. *Then*

$$\Gamma^j L_{\widetilde{\mathrm{U}}(r,s)}(\mu) = \begin{cases} \tau^{\lambda}_{\widetilde{\mathrm{U}}(r+s)} & \text{if } j = rs - (r - p)(s - q); \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

Moreover, there is a unique irreducible $\mathfrak{gl}(r,\mathbb{C})\oplus\mathfrak{gl}(s,\mathbb{C})$-submodule $W_{p,q}$ in $\bigwedge\mathfrak{g}/\mathfrak{k}$ with highest weight $2\rho(\mathfrak{u}\cap\mathfrak{p})$ (see. (2.20)). For any irreducible submodule $W\subset\bigwedge\mathfrak{g}/\mathfrak{k}$,

$$\Gamma_W L_{\widetilde{\mathrm{U}}(r,s)}(\mu) = \begin{cases} \Gamma^{rs-(r-p)(s-q)} L_{\widetilde{\mathrm{U}}(r,s)}(\mu) & \text{if } W = W_{p,q}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use array of integers present weights under the standard root systems. Let

$$\rho_m = (\frac{m-1}{2}, \frac{m-3}{2}, \cdots, \frac{-m+1}{2})$$

be the half sum of positive roots in $\mathfrak{gl}(m,\mathbb{C})$. Denote $\mathbf{a}[j_1 : j_2]$ the subarray of $\mathbf{a}$ consisting entries with index from j_1 to j_2 (if $j_2 < j_1$ the order is reversed). Let $\mathbf{a} = (a_1, \cdots, a_p)$ and $\mathbf{b} = (b_1, \cdots, b_q)$. Then infinitesimal character of $L_{\widetilde{\mathrm{U}}(r,s)}(\mu)$, via Harish-Chandra morphism, is

$$\begin{aligned}\chi :=&(\rho_{n-m}[n-m : s-q+1], a[p:1] + \rho_m[p:1] + \frac{r-s}{2}, \\ &- b[1:q] + \frac{r-s}{2} + \rho_m[m:p+1], \rho_{n-m}[s-q:1])\end{aligned}$$

Under Weyl group translation, infinitesimal character χ is conjugate to,

$$(\mu + \frac{r-s}{2} + \rho_m, \rho_{n-m}),$$

which could be obtained directly from the correspondence of infinitesimal characters for dual pairs (c.f. [Prz96]). Note that an unitary representation has non-zero cohomology only if its infinitesimal character is regular by Theorem 24 of Vogan-Zuckerman. Therefore $\Gamma^q L_{\widetilde{\mathrm{U}}(r,s)}(\mu)$ is non-zero only if

$$a_p \geq s-q \geq 0 \quad \text{and} \quad b_q \geq r-p \geq 0.$$

One may prove the lemma by Enright-Wallach's work[EW80]. But I prefer to apply Vogan-Zuckerman's theorem 23.

Let $x = (\underbrace{p, \cdots, 1}_{p}, 0, \cdots, 0, \underbrace{-1, \cdots, -q}_{q}) \in \mathfrak{t}$, define θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ by x, i.e.

$$\begin{aligned}\mathfrak{l} :=& Z_{\mathfrak{g}}(x) = \mathfrak{gl}(1,\mathbb{C})^p \oplus \mathfrak{gl}(r-p+s-q,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C})^q, \\ \mathfrak{u} :=& \sum_{\alpha(x)>0} \mathfrak{g}_\alpha.\end{aligned}$$

Now

$$
\begin{aligned}
R =& \dim \mathfrak{u}\cap\mathfrak{p} = rs-(r-p)(s-q),\\
2\rho(\mathfrak{u}\cap\mathfrak{p}) =&(\underbrace{s-q,\cdots,s-q}_{p},0,\cdots,0,\underbrace{-(r-p),\cdots,-(r-p)}_{q})\\
&+(\underbrace{q,\cdots,q}_{r},\underbrace{-p,\cdots,-p}_{s}).
\end{aligned}
\tag{2.20}
$$

Clearly λ is an admissible character (c.f. Definition 20). Since $\lambda + 2\rho(\mathfrak{u}\cap\mathfrak{p})$ is the highest weight of τ and $\lambda+\rho(\mathfrak{u}\cap\mathfrak{p})$ is the infinitesimal character of $L(\mu)$,

$$L_{\widetilde{\mathrm{U}}(r,s)}(\mu)\cong A_{\mathfrak{q}}(\lambda)$$

by Proposition 6.1 [VZ84] and $L_{\widetilde{U}(r,s)}(\mu)$ is unitarizable.

Now $\mathfrak{l}\cap\mathfrak{p}\cong\mathbb{C}^{r-p}\otimes\mathbb{C}^{s-q}$ as $\mathfrak{l}\cap\mathfrak{k}\cong\mathfrak{gl}(1,\mathbb{C})^p\oplus\mathfrak{gl}(r-p,\mathbb{C})\otimes\mathfrak{gl}(s-q,\mathbb{C})\oplus\mathfrak{gl}(1,\mathbb{C})^q$ module, where $\mathfrak{gl}(1,\mathbb{C})$ act trivially; $\mathfrak{gl}(r-p,\mathbb{C})$ and $\mathfrak{gl}(s-q,\mathbb{C})$ act by standard representation. The skew-duality (c.f. [How95, Theorem 4.1]) implies that, as $\mathfrak{l}\cap\mathfrak{k}$-module

$$\bigwedge^j\mathfrak{l}\cap\mathfrak{p}=\sum_{|D|=j}\tau^D_{\mathfrak{gl}(r-p,\mathbb{C})}\otimes\tau^{D^\top}_{\mathfrak{gl}(s-q,\mathbb{C})},$$

where D ranges over all Young digrams of size j with at most $r-p$ rows and at most $s-q$ columns, and $D^\top$ is the transpose of D.

Let F be the representation of $\mathfrak{g}$ with highest weight λ, then F^* has highest weight $-w_0\lambda$, where w_0 is the longest element in the Weyl group of $\mathfrak{g}$. So F^* has lowest weight $-\lambda$. Hence

$$
\begin{aligned}
H^j(\mathfrak{g},\mathfrak{k},L_{\widetilde{U}(r,s)}(\mu)\otimes F^*) =&\mathrm{Hom}_{\mathfrak{k}}(\bigwedge^j\mathfrak{p},L_{\widetilde{U}(r,s)}(\mu)\otimes F^*)\\
=&\mathrm{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\bigwedge^{j-R}\mathfrak{l}\cap\mathfrak{p},\mathbb{C})=\begin{cases}\mathbb{C} & \text{if } j=R,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

The $\mathfrak{k}$-module with highest weight $2\rho(\mathfrak{u}\cap\mathfrak{p})$ is multiplicity one in $\bigwedge\mathfrak{p}$. Let $W_{p,q}$ be such $\mathfrak{k}$-module, it is generated by the one-dimensional $\mathfrak{l}\cap\mathfrak{k}$-module $\bigwedge^R\mathfrak{u}\cap\mathfrak{p}\subset\bigwedge^R\mathfrak{p}$. Now, by the proof of Proposition 3.6 in [VZ84],

$$\mathrm{Hom}_{\mathfrak{k}}(\bigwedge^R\mathfrak{p},L_{\widetilde{U}(r,s)}(\mu)\otimes F^*)=\mathrm{Hom}_{\mathfrak{k}}(W_{p,q},L_{\widetilde{U}(r,s)}(\mu)\otimes F^*)=\mathbb{C}.$$

This finish the proof of the lemma. □

2.5 Invariants of representations

Let G be a real reductive Lie group. Let $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ under certain Cartan involution and K be the corresponding maximal compact subgroup of G. Let $G_{\mathbb{C}}$ (resp. $K_{\mathbb{C}}$) be the complexification of G (resp. K). Although some invariants could be defined in general setting, we will only consider $(\mathfrak{g}, K)$-modules of finite length. We follow Vogan's paper [Vog91].

We review some commutative algebra (c.f. Section 6 [MR89]) first. Let R be a commutative Noetherian ring and M be a finitely generated R-module. Define following invariants of M.

Associated variety

$$\mathcal{V}(M) = \{\, P \in \operatorname{Spec} R \mid P \supset \operatorname{Ann}_R(M) \,\}\,.$$

Support Let M_P be the localization of M at prime $P \subset R$.

$$\operatorname{Supp}(M) = \{\, P \in \operatorname{Spec}(R) \mid M_P \neq 0 \,\}\,.$$

Set of associated primes

$$\begin{aligned}\operatorname{Ass}(M) &= \{\, P \in \operatorname{Spec} R \mid \exists\, x \in M \text{ s.t. } \operatorname{Ann}_R(m) = P \,\} \\ &= \{\, P \in \operatorname{Spec} R \mid M \text{ contains a submodule isomorphic to } R/P \,\}\,.\end{aligned}$$

In our case, we have

$$\mathcal{V}(M) = \operatorname{Supp}(M) \supset \operatorname{Ass}(M).$$

For any ideal I of R, $\mathcal{V}(I)$ is the closed subscheme of $\operatorname{Spec} R$ defined by I. $\operatorname{Ass}(M)$ is a finite subset of $\operatorname{Supp}(M)$ and the set of minimal elements in $\operatorname{Ass}(M)$ and $\operatorname{Supp}(M)$ coincide. A prime in $\operatorname{Ass}(M)$ is called a *isolated prime* if it is minimal under inclusion relation. Call primes in $\operatorname{Ass}(M)$ which are not isolated associated primes *embedded primes* . Let $P_1, \cdots, P_r$ be the set of isolated associated primes of M. Then the varieties $\mathcal{V}(P_i)$ defined by P_i form the set of irreducible components of $\mathcal{V}(M)$. In particular,

$$\mathcal{V}(M) = \bigcup_i \mathcal{V}(P_i). \tag{2.21}$$

Note that M is Noetherian, so there exist a finite filtration $M_1 \subset \cdots \subset M_l = M$ such that $M_j/M_{j-1} \cong R/P$ for some $P \in \operatorname{Spec} R$. For every isolated associated prime $P \in \operatorname{Ass}(M)$ define the multiplicity of M at P to be

$$m(P,M) := \#\{\, j \mid M_j/M_{j-1} \cong R/P \,\}\,. \tag{2.22}$$

In fact, for any $P \in \operatorname{Spec} R$, define $m(P,M)$ to be the length of M_P as R_P-module, which is consistent with (2.22) if P is a minimal prime. So $m(P,M)$ is independent of the choice of filtrations.

We look the definition geometrically. Weakening the the requirement of the filtration, now, we only require M_j/M_{j-1} is generically reduced[11] [12] along prime P. Let $\mathscr{M}_j$ be the associated coherent sheave of M_j on $\operatorname{Spec} R$. Then there is a dense open set $U \subset \mathcal{V}(P) = \operatorname{Spec}(R/P)$ such that $(\mathscr{M}_j/\mathscr{M}_{j-1})|_U$ is free and $m(P,M)$ is equal to the sum of ranks of fibers in U, i.e.

$$m(P,M) = \sum_j \dim_{R/I_\lambda} M_j/(\mathcal{I}_\lambda M_j + M_{j-1}) \tag{2.23}$$

where $\lambda(= \mathcal{I}_\lambda) \in U$ is a closed point (maximal ideal).

Define the *characteristic cycle* of M

$$\operatorname{Ch}(M) = \sum_{i=1}^{r} m(P_i, M)\mathcal{V}(P_i), \tag{2.24}$$

which is a finer invariant than $\mathcal{V}(M)$ by (2.21).

Let $\mathcal{U}_j(\mathfrak{g})$ be the linear subset of $\mathcal{U}(\mathfrak{g})$ generated by $X_1X_2\cdots X_k$ where $X_1,\cdots,X_k \in \mathfrak{g}$ and $k \le j$. This gives a natural filtration of $\mathcal{U}(\mathfrak{g})$. Let $\operatorname{Gr}^j\mathcal{U}(\mathfrak{g}) = \mathcal{U}_j(\mathfrak{g})/\mathcal{U}_{j-1}(\mathfrak{g})$, then $\operatorname{Gr}\mathcal{U}(\mathfrak{g}) = \bigoplus_j \operatorname{Gr}^j\mathcal{U}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})$. Let $\sigma\colon \mathcal{U}(\mathfrak{g}) \to \operatorname{Gr}\mathcal{U}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g})$ be the symbol map and identify $\mathcal{S}(\mathfrak{g})$ with the polynomial ring on $\mathfrak{g}^*$. For a two sided ideal $\mathcal{I}$ in $\mathcal{U}(\mathfrak{g})$, define

$$\mathcal{V}(\mathcal{I}) := \mathcal{V}(\operatorname{Gr}\mathcal{I}) = \{\, x \in \mathfrak{g}^* \mid \sigma(u)(x) = 0 \ \forall u \in \mathcal{I} \,\}\,.$$

Now let V be a $(\mathfrak{g}, K)$-module of finite length. It is also an $\mathcal{U}(\mathfrak{g})$-module.

Definition 26. Define the *complex associated variety* of V to be

$$\mathcal{V}_{\mathbb{C}}(V) = \mathcal{V}(\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})}(V)) \subset \mathfrak{g}^*. \tag{2.25}$$

When V is an irreducible $\mathfrak{g}$-module, a deep theorem of Borho-Brylinski [BB85] and Joseph[Jos85] says that $\mathcal{V}_{\mathbb{C}}(V)$ is the closure of a single nilpotent

[11] Reduced on an open dense subset of $\mathcal{V}(P)$.

[12] In fact, all modules we studied are reduced long P. So they can be view as module over R/P.

coadjoint $G_{\mathbb{C}}$-orbit in $\mathfrak{g}^*$.

Now we can define associated variety and associated cycle of a $(\mathfrak{g}, K)$-module. A filtration $V_j \subset V_{j+1} \subset \cdots \subset V$ of $(\mathfrak{g}, K)$-module V is called *good* if it satisfies

$$\begin{aligned} \mathcal{U}_p(\mathfrak{g}) \cdot V_q \subset & V_{p+q} \\ K \cdot V_n \subset & V_n \\ V_{-n} = & 0 \quad \text{for } n \text{ sufficiently large;} \\ \bigcup_n V_n = & V; \\ \dim V_n < & \infty; \\ \mathcal{U}_p(\mathfrak{g}) \cdot V_q = & V_{p+q} \quad \text{for all } q \text{ sufficiently large and all } p \geq 0 \end{aligned} \tag{2.26}$$

Under a good filtration, $\operatorname{Gr} V := \bigoplus V_n/V_{n-1}$ is finite generated $\mathcal{S}(\mathfrak{g})$-module with equivariant $K_{\mathbb{C}}$-action. On the other hand, a $(\mathfrak{g}, K)$-module have a good filtration if and only if it is finitely generated by definition. Now we always assume filtrations are good.

Since $\mathfrak{k}$ act on $\operatorname{Gr} V$ trivially, $\operatorname{Supp}(\operatorname{Gr} V) \subset (\mathfrak{g}/\mathfrak{k})^*$.

Definition 27. View $\operatorname{Gr} V$ as an $\mathcal{S}(\mathfrak{p})$-module, define the *associated variety* of V to be

$$\mathcal{V}(V) = \operatorname{Supp}(\operatorname{Gr} V) \subset \mathfrak{p}^* \subset \mathfrak{g}^*.$$

Here we identify $(\mathfrak{g}/\mathfrak{k})^* \subset \mathfrak{g}^*$ with $\mathfrak{p}^*$ via Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Define the *nilpotent cone* in $\mathfrak{g}^*$ to be

$$\mathcal{N} := \{\, x \in \mathfrak{g}^* \mid p(x) = 0, \text{for all } p \in \mathcal{S}^+(\mathfrak{g})^G \,\}$$

where $\mathcal{S}^+(\mathfrak{g})^G$ is the set of $G_{\mathbb{C}}$-invariant polynomials on $\mathfrak{g}^*$ without constant term. Let $\{\, \mathcal{O}_1, \cdots, \mathcal{O}_r \,\}$ be the set of open $K_{\mathbb{C}}$-orbits in $\mathcal{V}(V)$. We have

$$\mathcal{V}(V) = \bigcup_{j=1}^{r} \overline{\mathcal{O}}_j \subset \mathcal{N} \cap \mathfrak{p}^*,$$

Although the closure of each $K_{\mathbb{C}}$-orbit $\mathcal{O}_j$ may be reducible, the multiplicity along the irreducible components of $\overline{\mathcal{O}}_j$ are the same, since $\operatorname{Gr} V$ is $K_{\mathbb{C}}$-equivariant. Let $m(\mathcal{O}_i, V)$ be the common multiplicity.

Definition 28. Define the *associated cycle* of V to be

$$\operatorname{AC}(V) := \operatorname{Ch}(\operatorname{Gr} V) = \sum_i m(\mathcal{O}_i, V)[\overline{\mathcal{O}_i}].$$

Following theorem proved by Vogan shows the relationship between the complex associated variety and associated variety.

Theorem 29 (Theorem 8.4 [Vog91]). *Let V be an irreducible $(\mathfrak{g}, K)$-module. Let $\mathcal{O}$ be the open dense nilpotent $G_{\mathbb{C}}$-orbit in $\mathcal{V}_{\mathbb{C}}(V)$. Then*

(i) $\mathcal{V}(V) \subset \mathcal{V}_{\mathbb{C}}(V) \cap \mathfrak{p}^*$;

(ii) $\mathcal{O} \cap \mathfrak{p}^*$ *is a finite union of* $K_{\mathbb{C}}$*-orbit* $\mathcal{O}_1, \cdots \mathcal{O}_s$*, each orbit has dimension* $\frac{1}{2} \dim \mathcal{O}$;

(iii) Some of the $\mathcal{O}_i$ *are contained in* $\mathcal{V}(V)$*; they are precisely the* $K_{\mathbb{C}}$*-orbits of maximal dimension in* $\mathcal{V}(V)$*, i.e.*

$$\mathcal{V}(V) = \bigcup_{\mathcal{O}_i \subset \mathcal{V}(V)} \overline{\mathcal{O}}_i.$$

Now we explain why these invariants reflect the "size" of a $(\mathfrak{g}, K)$-module. Define the *Gelfand-Kirillov dimension*

$$\dim V := \dim \operatorname{Gr} V = \dim \mathcal{V}(V) = \frac{1}{2} \dim \mathcal{V}_{\mathbb{C}}(V).$$

Now there is a unique polynomial $Q(t)$ such that following formal series[13], called *Hilbert-Poincare series* of $\operatorname{Gr} V$ satisfies

$$P(V; t) = \sum_{n=0}^{\infty} (\dim V_n / V_{n-1}) t^n = \frac{Q(t)}{(1-t)^{\dim V}} \quad \text{and} \quad Q(1) \neq 0.$$

Moreover $\deg V := Q(1)$ is an integer called its *Bernstein degree*. The Bernstein degree could be recovered from associated cycle:

$$\deg V = \sum_i m(\mathcal{O}_i, V) \deg \overline{\mathcal{O}_i},$$

where $\deg \overline{\mathcal{O}_i}$ is the degree of $\mathcal{O}_i$ as projective sub-variety of $\mathrm{P}\mathfrak{p}^*$.

To capture the information on K-spectrums, we define the isotropic representations. Let $M := \operatorname{Gr} V$, Vogan shows (c.f. Lemma 2.11 [Vog91]) that M always have a finite filtration $\{ M_j \}$ by $(\mathcal{S}(\mathfrak{g}), K)$-submodules such that each M_j / M_{j-1} is generically reduced along every minimal prime in $\mathcal{V}(M)$. Let λ be a element in the open orbit $\mathcal{O}$ of $\mathcal{V}(V)$. The *isotropic subgroup* $K_\lambda = \mathrm{Stab}_{K_{\mathbb{C}}}(\lambda)$ is the stabilizer of P. Now the fiber of $\mathscr{M}_j / \mathscr{M}_{j-1}$

[13] It reflects the growth of dimension with respect to degree.

at λ,

$$M_j/(\mathcal{I}_\lambda M_j + M_{j-1}),$$

is a finite dimensional rational K_λ-module.

Definition 30. Define the *isotropic representation* of M at λ to be the genuine virtual character

$$\chi(\lambda, M) = \sum_j M_j/(\mathcal{I}_\lambda M_j + M_{j-1}).$$

Clearly,

$$\dim \chi(\lambda, M) = m(\mathcal{O}, V).$$

Although, Vogan has proved more general statement, to motivate the further discussion, we only need the following form.

Theorem 31 (Theorem 4.6, Theorem 4.11 [Vog91])**.** *Let V be an irreducible $(\mathfrak{g}, K)$-module and $\mathcal{O} = K_{\mathbb{C}} \cdot \lambda$ be a $K_{\mathbb{C}}$ orbit whose closure contains an irreducible component of $\mathcal{V}(V)$. Then either*

a) $\mathcal{V}(V) = \overline{\mathcal{O}}$ or

b) $\partial\mathcal{O}$ has codimension one in $\overline{\mathcal{O}}$.

If $\partial\mathcal{O}$ has codimension at least 2 in $\overline{\mathcal{O}}$, let V_χ be a completely reducible representation of K_λ which has character $\chi(\lambda, X)$. Then there is a finitely generated $(\mathcal{S}(\mathfrak{g}), K)$-module Q supported on $\partial\mathcal{O}$ such that

$$V = \mathrm{Ind}_{K_\lambda}^{K_{\mathbb{C}}}(V_\chi) - Q$$

as a virtual representation of $K_{\mathbb{C}}$.

This theorem suggests that up to an error Q the K-spectrum of V is controlled by its isotropic representation. Furthermore, Vogan defined a notion of "admissible data" corresponding to certain nilpotent $K_{\mathbb{C}}$-orbits and then conjectured that for each such data there is an unipotent representations attached to it such that its K-spectrum could be recovered by the admissible data.

Now we state the conjecture more precisely. For a nilpotent element $\lambda \in \mathfrak{p}^*$, K_λ is the stabilizer of λ in $K_{\mathbb{C}}$. Let $\mathfrak{k}_\lambda = \mathrm{Lie}(K_\lambda)_{\mathbb{C}}$. An rational representation χ of K_λ is called admissible if restricted on the connected component of K_λ,

$$\chi(\exp(x)) = \det(\mathrm{Ad}(\exp(x/2)|_{(\mathfrak{k}/\mathfrak{k}(\lambda))^*})) \cdot \mathrm{id} \quad \forall x \in \mathfrak{k}(\lambda).$$

The pair (λ, χ) is called *admissible data*. An $K_{\mathbb{C}}$-orbit in $\mathfrak{p}^*$ is called an *admissible orbit* if there exist an admissible data attached to it.

Conjecture 32 (Conjecture 12.1 [Vog91])**.** *Let $\mathcal{O}$ be a $G_{\mathbb{C}}$ nilpotent orbit in $\mathfrak{g}^*$ such that $\partial\mathcal{O}$ has codimension at least 4 in $\overline{\mathcal{O}}$. Let V be an irreducible unipotent $(\mathfrak{g}, K)$-module attached to $\mathcal{O}$. Then there exist admissible data (λ, χ) with $\lambda \in \mathcal{O} \cap \mathfrak{p}^*$ such that, as a representation of $K_{\mathbb{C}}$*

$$V \cong \operatorname{Ind}_{K_\lambda}^{K_{\mathbb{C}}}(\chi). \tag{2.27}$$

By Theorem 31, the associated cycle of V should be $\dim \chi \cdot [\overline{K_{\mathbb{C}} \cdot \lambda}]$. However, the notion "unipotent representation" is not properly defined. So, instead of trying to prove above conjecture, we show some families of representations satisfied equation (2.27).

2.6 Representations of algebraic groups

Now we review some basic facts about the representations of algebraic groups. Most of them are natural and similar to the corresponding theory of compact Lie groups. However, the proof of some theorems are not trivial. The author think the textbooks of Borel[Bor91], Grosshans[Gro97], Popov et al. [PV94], Jantzen[Jan87] and Mumford et al.[MFK94] are very useful references. All groups below are assume to be linear algebraic group scheme over a algebraically closed field k and all group action on scheme are assume to be morphisms. Since all schemes appeared in our application are certain subset of vector spaces over $\mathbb{C}$, we will assume all schemes are reduced, separated schemes and of finite type over an algebraically closed field k from now on. While, we would like to point out that, in fact, most theorems are proved for (reduced) schemes over (even non-algebraically closed) field k. For any algebraic group scheme G, a *rational representation* of G on vector space V is a group homomorphism $G \to \mathrm{GL}(V)$ such that it is also a morphism between schemes. We will assume all G-modules are rational.

2.6.1 Quotients

Definition 33 (Definition 0.5-0.6 [MFK94], Section 4 [PV94])**.** Let G be a algebraic group over k, X be an algebraic variety over k and G act on X rationally.

(a) An algebraic variety Y together with a morphism $\pi_Y : X \to Y$ is called *categorical quotient* for the acton of G on X if it satisfies following universal property: for any morphism $\pi_Z\colon X \to Z$, which is constant on the orbits of G, there is a unique morphism $\alpha\colon Y \to Z$ such that $\pi_Z = \alpha \circ \pi_Y$.

(b) (Y, π_Y) in (a) is called *geometric quotient* if it further satisfies:

(i) the morphism π_Y is surjective;

(ii) the morphism π_Y is open;

(iii) fibers are precisely the G-orbit.

(iv) for all open set $U \subset Y$, the morphism $\pi_Y^*\colon k[U] \to k[\pi_Y^{-1}(U)]^G$ is an isomorphism.

From now on, we assume all groups are linear algebraic groups over a fixed algebraic closed field k, i.e. closed subgroups of the general linear groups. For applications here, $k = \mathbb{C}$.

When X is an affine variety and G is reductive, $k[X]^G$ is finitely generated. Define the *affine quotient* of X to be the affine variety $X/G := \operatorname{Spec} k[X]^G$ and the *affine quotient map* $\pi_{X/G}\colon X \to X/G$ is given by embedding $k[X]^G \hookrightarrow k[X]$.

Theorem 34 ([PV94, Section 4.4]). *(i) The pair $(X/G, \pi_{X/G})$ is a categorical quotient for the action of G on X.*

(ii) For any open subset $U \subset X/G$ is a categorical quotient for the action of G on $\pi_{X/G}^{-1}(U)$.

(iii) For G-invariant closed set $Z \in X$, $\pi_{X/G}(Z)$ is closed in X/G.

(iv) The pair $(X/G, \pi_{X/G})$ is a geometric quotient if and only if all orbits of G in X are closed.

2.6.2 Homogenous spaces

For a closed subgroup H in G, the homogenous space G/H is the geometric quotient for the right translation of H on G.

The homogenous space G/H is always quasi-projective. Following theorem listed some sufficient conditions for it to be affine.

Theorem 35. *When H is reductive, G/H is affine. When G is a reductive groups, the homogeneous space G/H is affine if and only if H is reductive.*

In our application, G/H is quasi-affine, i.e. a open subset of an affine variety. Such H is called a *observable subgroup*. In fact, there will be a rational representation V (some spaces of matrices) of G and H will be the stabilizer of a point $v \in V$ (c.f. Chapter 1[Gro97] for equivalent definitions and properties of *observable subgroups*. Now G/H is isomorphic to orbit $G \cdot v \subset V$, which is an opens dense subset in it closure $\overline{G \cdot v}$.

2.6.3 Induced modules and their associated sheaves

In this section, let H be a closed subgroup of a linear algebraic group G.

Definition 36. For any H-module V, define the induced module of V to be the space of H-invariants in $k[G] \otimes V$, i.e.

$$\mathrm{Ind}_H^G V := (k[G] \otimes V)^H ,$$

where H act by right translation and G act by left translation on $k[G]$.

By definition, Ind_H^G is a functor from the category of rational H-modules to the category of rational G-module. Moreover, it also has a $k[G]^H$-module structure with $k[G]^H$ act on $\mathrm{Ind}_H^G V$ by multiplication on the factor $k[G]$. Therefore, we can view $\mathrm{Ind}_H^G W$ as a $(k[G]^H, G)$-module.

Theorem 37 (Section 6 [Gro97])**.** *Let V be a H-module, W be a G-module and $W|_H$ be the restriction of W to H.*

*(a) (*Frobenius Reciprocity*)*

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G V) \cong \mathrm{Hom}_H(W|_H, V).$$

*(b) (*Induction by stage*) Let L be closed subgroups of G such that $H < L < G$. Then, there is a $k[G]^L \subset k[G]^H$-equivariant isomorphism of G-modules:*

$$\mathrm{Ind}_L^G(\mathrm{Ind}_H^L V) \cong \mathrm{Ind}_H^G V.$$

(c) $k[G]^H \otimes W \cong \mathrm{Ind}_H^G(W|_H)$.

(d) $\mathrm{Ind}_G^G W \cong W$ as G-module.

*(e) (*tensor identity*) There is a $(k[G]^H, G)$-equivariant isomorphism*

$$(\mathrm{Ind}_H^G V) \otimes W \cong \mathrm{Ind}_H^G(V \otimes W|_H).$$

*(f) (*the transfer principle*) $(k[G] \otimes W)^G \cong W$ as G-module. The space of invariants is taking with respect to the diagonal G-action in which G acton $k[G]$ by left translation. And the resulting module inherent the G-module structure from right translation.*

(g) The induction functor is left exact, i.e. for exact sequence of H-module $0 \to V_1 \to V_2 \to V_3$, we have

$$0 \to \mathrm{Ind}_H^G V_1 \to \mathrm{Ind}_H^G V_2 \to \mathrm{Ind}_H^G V_3.$$

(h) The quotient space G/H is affine if and only if the induction functor is exact.

For a H-module V, we can associate a quasi-coherent sheaf $\mathscr{L}$ on G/H such that,for any affine open set $U \subset G/H$,

$$\mathscr{L}_V^{G/H}(U) = \left(k[\pi_{G/H}^{-1}(U)] \otimes V\right)^H.$$

When G/H and V is fixed, we write $\mathscr{L}_V^{G/H}$ simply by $\mathscr{L}_V$ or $\mathscr{L}$. Then the induction functor is same as the global section functor, i.e.

$$\mathrm{Ind}_H^G V = H^0(G/H, \mathscr{L}_V). \tag{2.28}$$

Moreover, when V is finite dimensional H-module, $\mathscr{L}_V$ is locally free of finite rank, therefore is a coherent sheaf. In fact, for any affine open set $U \subset G/H$, $\mathscr{L}_V(U) \cong k[U] \otimes V$ (c.f. [CPS83, Corollary 2.10], or [Jan87, Section 5.9]).

Note that part (h) of Theorem 37 is a consequence of (2.28). Combining Theorem 35 and Theorem 37 (h), the induction functor is exact if G and H are reductive.

We should point out that, we also can view the quasi-coherent sheaf $\mathscr{L}$ as the sheaf of regular sections of the vector bundle $p\colon G \times^H V \to G/H$, where $G \times^H V$ is the (geometric) quotient of $G \times V$ by diagonal H action. And than, for any open subset U of G/H, $\mathscr{L}(U) \cong \Gamma(U, G \times^H V)$ where $\Gamma(U, G \times^H V)$ denote the set of morphisms[14] s from U to $G \times^H V$ such that $p \circ s = \mathrm{id}_U$.

Let X be a variety with G action and $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_X$-module. Let $\mathbb{V}(\mathscr{E}) = \mathrm{Spec}\, S(\mathscr{E})$ be the vector bundle associated to $\mathscr{E}$, where $S(\mathscr{E})$ is the symmetric algebra of $\mathscr{E}$ over $\mathscr{O}_X$. A action G action on $\mathbb{V}(\mathscr{E})$, $G \times \mathbb{V}(\mathscr{E}) \to \mathbb{V}(\mathscr{E})$, is given by an $\mathcal{O}_X$-algebra homomorphism $S(\mathscr{E}) \to p_*(\mathscr{O}_G \otimes_k S(\mathscr{E}))$, where $p\colon G \times X \to X$ defines the action of G on X.

Definition 38 ([CPS83, Definition (2.3)])**.** A *quasi-coherent (X, G)-module* is a quasi-coherent $\mathscr{O}_X$-module $\mathscr{E}$ equipped with a left action of G on $\mathbb{V}(\mathscr{E})$

[14] May view as regular sections.

such that the $\mathcal{O}_X$-algebra homomorphism $S(\mathscr{E}) \to p_*(\mathscr{O}_G \otimes_k S(\mathscr{E}))$ carries $\mathscr{E}$ into $p_*(\mathscr{O}_G \otimes_k \mathscr{E})$.

We will use following equivalence of categories frequently later.

Theorem 39 ([CPS83, Theorem 2.7] imprimitivity theorem)**.** *Let H be a closed subgroup of an affine algebraic k-group scheme G. Let $X = G/H$ and $e = H/H \in G/H$. Then*

$$V \mapsto \mathscr{L}_V^{G/H} \quad \textit{and} \quad \mathscr{E} \mapsto k \otimes_{\mathscr{O}_{X,e}} \mathscr{E}_e$$

define an equivalence between the category of rational H-modules and that of quasi-coherent (X, G)-modules. In particular, every quasi-coherent (X, G)-module is induced.

Theorem 40 ([CPS83, Theorem 4.4] Mackey decomposition theorem)**.** *Let L be a closed subgroup of G. Suppose the L-orbit $\Omega \subset G/H$ of $e = H/H$ is open in G/H.*

(a) If $\Omega = G/H$, then, as L-module,

$$H^n(G/H, \mathscr{L}_V^{G/H}) \cong H^n(L/(L \cap H), \mathscr{L}_V^{L/(L\cap H)}), \quad \forall\, n \geq 0. \tag{2.29}$$

In particular, for $n = 0$, we have

$$\mathrm{Ind}_H^G V \cong \mathrm{Ind}_{L\cap H}^L V. \tag{2.30}$$

(b) Otherwise, let $d := \mathrm{codim}(G/H - \Omega)$ be the codimension of the boundary of Ω. Then (2.29) hold for $0 \leq n \leq d - 1$. In particular, (2.30) holds when $d \geq 2$.

Chapter 3

Derived functor modules of local theta lifts

3.1 Introduction

In this Chapter, we will study some families of small representations obtained by taking derived functors on certain theta lifts. The main objective is to show these representations are again certain theta lifts. This project is motivated by the work of Wallach and Zhu [WZ04].

Let $\mathfrak{g}_0$ be the Lie algebra of a real reductive group G, K be a maximal compact subgroup of G. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. We focus on admissible $(\mathfrak{g}, K)$-modules. Let $\mathscr{C}(\mathfrak{g}, K)$ be the category of $(\mathfrak{g}, K)$-modules. Let M be a subgroup of K. Zuckerman functor $\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}$ is the functor from $\mathscr{C}(\mathfrak{g}, M)$ to $\mathscr{C}(\mathfrak{g}, K)$ right adjoint to the forgetful functor. This functor is only left exact and usually will be zero on modules in $\mathscr{C}(\mathfrak{g}, M)$. Its right derived functor $R^j\Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}$ constructs interesting representations.

Using derived functors, we can transfer representations between different real forms of a complex reductive group as following. Let $\mathfrak{g}$ be a complex Lie algebra of complex reductive group $G_{\mathbb{C}}$; $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two different real forms of $\mathfrak{g}$; θ_1 and σ_1(resp. θ_2 and σ_2) be Cartan involution and complex conjugation on $\mathfrak{g}$ for $\mathfrak{g}_1$(resp. $\mathfrak{g}_2$); G_i be the maximal subgroups of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_i$; K_i be corresponding maximal compact subgroup and $\mathfrak{k}_i = \mathrm{Lie}(K_i)_{\mathbb{C}}$. Assume all these involutions are commute to each other, i.e.

$$\sigma_1\sigma_2 = \sigma_2\sigma_1, \quad \theta_1\theta_2 = \theta_2\theta_1, \quad \theta_1\sigma_2 = \sigma_2\theta_1, \quad \theta_2\sigma_1 = \sigma_1\theta_2. \tag{3.1}$$

The diagram, see Figure 3.1, consisting of four diamonds may be helpful to understand the relationship between the algebras defined above (c.f. [WZ04]).

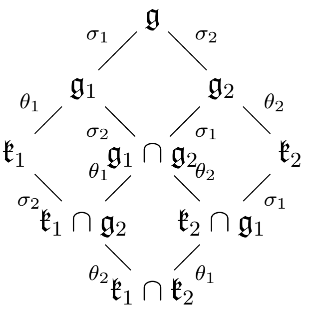


Figure 3.1: A diamond of Lie algebras

Let $M = K_1 \cap K_2$. Composing the forgetful functor

$$\mathcal{F}^{\mathfrak{g},M}_{\mathfrak{g},K_1} : \mathscr{C}(\mathfrak{g}, K_1) \to \mathscr{C}(\mathfrak{g}, M)$$

and derived functors

$$\mathrm{R}^j\Gamma^{\mathfrak{g},K_2}_{\mathfrak{g},M} : \mathscr{C}(\mathfrak{g}, M) \to \mathscr{C}(\mathfrak{g}, K_2),$$

produce a family of functors

$$\Gamma^j := \mathrm{R}^j\Gamma^{\mathfrak{g},K_2}_{\mathfrak{g},M} \circ \mathcal{F}^{\mathfrak{g},M}_{\mathfrak{g},K_1} : \mathscr{C}(\mathfrak{g}, K_1) \to \mathscr{C}(\mathfrak{g}, K_2).$$

This procedure construct $(\mathfrak{g}, K_2)$-modules (locally K_2- finite) from a $(\mathfrak{g}, K_1)$-module (locally K_1-finite). We call it *transfer of K-types*.

On the other hand, theta lifting also constructs representations for different real forms. Let G be a classical group in a real reductive dual pair $(G, G') \subset \mathrm{Sp}(W)$ where W is a real symplectic space. Choose a maximal compact subgroup U in $\mathrm{Sp} := \mathrm{Sp}(W)$ such that intersection with G and G' give their maximal compact subgroup K and K'. Let $\widetilde{\mathrm{Sp}}$ be the metaplectic cover of Sp. For any subgroup $E \subset \mathrm{Sp}$, let $\widetilde{E}$ be its inverse image in $\widetilde{\mathrm{Sp}}$ of the projection $\widetilde{\mathrm{Sp}} \to \mathrm{Sp}$, which is certain double cover of E. Let $\mathscr{Y}$ be the Harish-Chandra module (Fock model) of the oscillator representation ω of the metaplectic group $\widetilde{\mathrm{Sp}}$ with respect to its maximal compact subgroup $\widetilde{\mathrm{U}}$. For any subgroup E of Sp such that $K_E := E \cap \mathrm{U}$ is a maximal compact subgroup of E, let $\mathfrak{e}$ be the Lie algebra of E and $\mathscr{R}(\mathfrak{e}, \widetilde{K}_E; \mathscr{Y})$ be the infinitesimal equivalent classes of irreducible $(\mathfrak{e}, \widetilde{K}_E)$-modules which can be realized as a quotient of $\mathscr{Y}$. Howe [How89b] constructs a bijection $\theta \colon \mathscr{R}(\mathfrak{g}, \widetilde{K}; \mathscr{Y}) \to \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$. This map is called theta lifting. By abuse notion, we also call its inverse θ.

Now let G_1 and G_2 be two real forms of a classical complex Lie group $G_\mathbb{C}$ satisfying conditions defining Γ^j. We will exhibit some relationships between transfer of K-type (taking derived functor module) and theta lifting. The naive guess is that they "commute" with each other. More precisely, for dual pairs (G_i, G_i') $(i = 1, 2)$, we are seeking and some operations filling the gap in the following commutative diagram.

$$\begin{array}{ccccc}
\mathscr{R}(\mathfrak{g}', \widetilde{K}_1') & \xrightarrow{\theta} & \mathscr{R}(\mathfrak{g}, \widetilde{K}_1) & \hookrightarrow & \mathscr{C}(\mathfrak{g}, \widetilde{K}_1) \\
\Big\downarrow ? & & & & \Big\downarrow \Gamma^j \\
\mathscr{R}(\mathfrak{g}', \widetilde{K}_2') & \xrightarrow{\theta} & \mathscr{R}(\mathfrak{g}, \widetilde{K}_2) & \hookrightarrow & \mathscr{C}(\mathfrak{g}, \widetilde{K}_2)
\end{array}$$

However, the actual relationship is more subtle, as following example suggests.

Example 41 (Theorem 56). *Fix integers m, n, r, s such that $m \leq n = r+s$. For integer p, q such that $p + q = m$, let $\theta^{p,q}$ be the theta lifting map from $\mathrm{O}(p, q)$ to $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$. Let $\Gamma^j = R^j\Gamma^{\mathfrak{sp}(2n,\mathbb{C}),\widetilde{\mathrm{U}}(n)}_{\mathfrak{sp}(2n,\mathbb{C}),\widetilde{\mathrm{U}}(r)\cdot\widetilde{\mathrm{U}}(s)} \circ \mathcal{F}^{\mathfrak{sp}(2n,\mathbb{C}),\widetilde{\mathrm{U}}(r)\cdot\widetilde{\mathrm{U}}(s)}_{\mathfrak{sp}(2n,\mathbb{C}),\widetilde{\mathrm{U}}(n)}$. Then, as $(\mathfrak{sp}(2n, \mathbb{C}), \widetilde{\mathrm{U}}(n))$-module,*

$$\Gamma^j(\theta^{m,0}(\det{}^\epsilon)) = \bigoplus_{\substack{p \leq r, q \leq s \\ j = rs-(r-p)(s-q)}} \theta^{p,q}(\mathbf{1}^{\xi,\eta}). \tag{3.2}$$

Here $\xi \equiv \epsilon-(s-q), \eta \equiv \epsilon-(r-p) \pmod 2$, $\det^\epsilon$ is the trivial or determinate representation of $\mathrm{O}(p, q)$ depends on the parity of ϵ, $\mathbf{1}^{\xi,\eta}$ are characters of $\mathrm{O}(p, q)$ determined by $\mathbf{1}^{\xi,\eta}|_{\mathrm{O}(p)\times\mathrm{O}(q)} = \det^\xi \otimes \det^\eta$.

Wallach and Zhu [WZ04] conjectured equation (3.2) for $\epsilon = 0$ by K-spectrum comparison. The motivation of this study is to prove their conjecture. For similar results of other dual pairs, see Theorem 58 and Theorem 60.

For any $\rho \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$, denote its maximal Howe quotient by $\Theta(\rho)$. The following is our key theorem.

Theorem A. *Let G_1' and G_2' be two real forms of a classical complex lie group $G_\mathbb{C}'$ such that $(G_i, G_i')_{i=1,2}$ form reductive dual pairs. Let ρ_1 and ρ_2 be characters of $\widetilde{G}_1'$ and $\widetilde{G}_2'$ respectively; τ_1 be an irreducible $(\mathfrak{k}_2, \widetilde{K}_1 \cap \widetilde{K}_2)$-module; τ_2 be an irreducible $\widetilde{K}_2$-module. Suppose that:*

(a) $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(\rho_1) = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(\rho_2)$.

(b) There is $0 \neq T \in \mathrm{Hom}_{\mathfrak{k}_2, \widetilde{K}_1\cap\widetilde{K}_2}(\Theta(\rho_1), \tau_2)$, such that τ_2 occur both in the image of $\Gamma^j T \colon \Gamma^j\Theta(\rho_1) \to \Gamma^j V_{\tau_1}$ and $\Theta(\rho_2)$.

Then $(\mathfrak{g}, \widetilde{K}_2)$*-modules* $\Gamma^j\Theta(\rho_1)$ *and* $\Theta(\rho_2)$ *have an isomorphic irreducible subquotient with common* $\tilde{K}_2$*-type* τ_2*.*

Remark:

1. Theorem A implies that the K-spectrum and infinitesimal characters determined the derived functor modules of theta lifts of characters. On the other hand, if $\mathfrak{g}'$ is a semisimple Lie algebra, i.e. $\mathfrak{so}$ and $\mathfrak{sp}$, the only character of $\mathfrak{g}'$ is the trivial representation; therefore, condition (a) is always satisfied.

2. Example 41 is a consequence of Theorem A.

3. The K-spectrum of derived functor modules were already calculated in many cases, for example Enright et al.[EPWW85], Frajria [Fra91], Wallach [Wal94] and Wallach and Zhu [WZ04]. Their calculations will provide families of examples for Theorem A, although they are not originally presented in this way,.

The proof of above theorem combines following observations:

(a) Equation (3.6) on the Hecke-algebra actions, which may have potential usage beyond Theorem A.

(b) A result on the algebra of invariant differential operators which goes back to Helgason, and Lemma 45.

(c) The $\mathfrak{g}$-actions on the derived functor modules (3.8) is defined in a functorial way.

3.2 A space with $\mathcal{U}(\mathfrak{g})^H$ action

In this section, we will review the well known see-saw pair argument and exhibit an additional Hecke-algebra module structure on the multiplicity space.

Let G be a real reductive group and $K = K_G$ be a maximal compact subgroup of G. If H is a subgroup of G, we always assume that $K_H := H \cap K_G$ is a maximal compact subgroup of H. For every such Harish-Chandra pair $(\mathfrak{g}, K)$, $\mathscr{C}(\mathfrak{g}, K)$ denote the category of $(\mathfrak{g}, K)$-modules (not necessary admissible). For every $V \in \mathscr{C}(\mathfrak{g}, K_G)$, we can view it as an element in $\mathscr{C}(\mathfrak{h}, K_H)$ via forgetful functor.

Definition 42. Let V be an $(\mathfrak{g}, K_G)$-module and U be an irreducible $(\mathfrak{h}, K_H)$-module. Define

$$\Omega_{V,U} = V/\mathcal{N}_{V,U}, \qquad \text{where } \mathcal{N}_{V,U} = \bigcap_{T\in \mathrm{Hom}_{\mathfrak{h},K_H}(V,U)} \mathrm{Ker}\,(T).$$

Clearly, $\Omega_{V,U}$ is an $(\mathfrak{h}, K_H)$-module by definition. Additionally, $\Omega_{V,U}$ has an has a natural $\mathrm{Hom}_{\mathfrak{h},K_H}(V,V)$-action defined by

$$S(\overline{v}) = \overline{S(v)} \quad \forall\, \overline{v} \in V/\mathcal{N}_{V,U},\ S \in \mathrm{Hom}_{\mathfrak{h},K_H}(V,V),$$

where $\overline{v}$ denote the image of $v \in V$ in quotient $V/\mathcal{N}_{V,U}$. For all $T \in \mathrm{Hom}_{\mathfrak{h},K_H}(V,U)$, $T \circ S \in \mathrm{Hom}_{\mathfrak{h},K_H}(V,U)$. So the action is well defined, since for every $v \in \mathcal{N}_{V,U}$, $T \circ S(v) = 0$, i.e. $S(v) \in \mathcal{N}_{V,U}$. In particular, $\mathcal{U}(\mathfrak{g})^H$ act on $\Omega_{V,U}$ via $\mathcal{U}(\mathfrak{g})^H \to \mathrm{Hom}_{\mathfrak{h},K_H}(V,V)$.

By imitating a proof of Moeglin et al. [MVW87, Lemma III.4, Chapter 2], as $\mathrm{Hom}_{\mathfrak{h},K_H}(V,V) \times (\mathfrak{h}, K_H)$-module

$$\Omega_{V,U} \cong U \otimes U', \quad \text{where} \quad U' \cong \mathrm{Hom}_{\mathfrak{h},K_H}(U, \Omega_{V,U}),$$

$(\mathfrak{h}, K_H)$ act on U and $\mathcal{U}(\mathfrak{g})^H$ act on U' (by post-composition) (c.f. Lemma 5, Section 2.3.3).

Remark: This is a generalization of Howe's construction of maximal quotient:

1. When $V := \mathscr{Y}$ is a Fock model of the oscillator representation, $G = \widetilde{\mathrm{Sp}}$, $K = \widetilde{\mathrm{U}}'$ and (H, H') are reductive dual pairs in $\widetilde{\mathrm{Sp}}$, U' is the maximal Howe quotient $\Theta(U)$ of U. It is a $(\mathfrak{h}', K'_H)$-module, since H' commute with H.

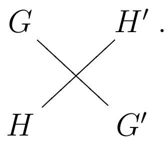
2. When $H := K$ is the maximal compact subgroup of G, V is an $(\mathfrak{g}, K)$-module, U is an irreducible K-module occur in V, $\Omega_{V,U} \cong U \otimes \mathrm{Hom}_K(U,V)$ isomorphic to the U-isotypic component of V. Later we will use a classical result of Hraish-Chandra and Lepowsky-McCollum [LM73] (c.f. Section 2.2) asserts that: the isomorphism class of irreducible admissible $(\mathfrak{g}, K)$-module V is determined by the $\mathcal{U}(\mathfrak{g})^K$ action on $\mathrm{Hom}_K(U,V)$, if $\mathrm{Hom}_K(U,V) \neq 0$. On the other hand, $\mathrm{Hom}_K(U, \Omega_{V,U})$ is naturally isomorphic to the dual of $\mathrm{Hom}_K(\Omega_{V,U}, U)$. Therefore it is equivalent to known $\mathcal{U}(\mathfrak{g})^K$ actions on the $\mathrm{Hom}_K(\Omega_{V,U}, U)$.

The following equation is the key property of $\Omega_{V,U}$:

$$\mathrm{Hom}_{\mathfrak{h},K_H}(V,U) \cong \mathrm{Hom}_{\mathfrak{h},K_H}(\Omega_{V,U}, U), \tag{3.3}$$

where $\mathrm{Hom}_{\mathfrak{h},K_H}(V,V)$ act on $\mathrm{Hom}_{\mathfrak{k},K_H}(V,U)$ by pre-composition and the isomorphism is $\mathrm{Hom}_{\mathfrak{h},K_H}(V,V)$-equivariant.

Equation 3.3 leads the well known see-saw pair argument. A pair of reductive dual pairs (G, G') and (H, H') in a symplectic group are called a *see-saw pair* if $H \leq G$ and (therefore) $H' \geq G$. It can be represented by following diagram, in which groups connected by straight lines form reductive dual pairs.



As in [How89b], we choose a maximal compact subgroup U in $\mathrm{Sp}(W)$ such that intersection with G, G', H and H' give their maximal compact subgroups. We fix the oscillator representation[1] ω and fix a Fock module $\mathscr{Y}$ with respect to U (c.f. Section 2.3.4). ,

For any reductive dual pair (G, G') and representation $\rho \in \mathscr{R}(\mathfrak{g}', \widetilde{K}_{G'}; \mathscr{Y})$, the space realizing ρ by V_ρ. Since V_ρ can be realized as a quotient of the oscillator representation ω, we have

$$\Omega_{\mathscr{Y}, V_\rho} \cong \Theta(\rho) \otimes V_\rho$$

where the $(\mathfrak{g}, \widetilde{K}_G)$-module $\Theta(\rho)$ is the maximal Howe quotient. Abuse of notation, Θ also denote the maximal quotient from H to H'.

Lemma 43. *For any $\tau \in \mathscr{R}(\mathfrak{h}, \widetilde{K}_H; \mathscr{Y})$ and $\rho \in \mathscr{R}(\mathfrak{g}', \widetilde{K}_{G'}; \mathscr{Y})$,*

$$\mathrm{Hom}_{\mathfrak{h}, \widetilde{K}_H}(\Theta(\rho), V_\tau) \cong \mathrm{Hom}_{(\mathfrak{h}, \widetilde{K}_H) \times (\mathfrak{g}', \widetilde{K}_{G'})}(\mathscr{Y}, V_\tau \otimes V_\rho) \cong \mathrm{Hom}_{\mathfrak{g}', \widetilde{K}_{G'}}(\Theta(\tau), V_\rho), \tag{3.4}$$

where the first isomorphism is $\mathcal{U}(\mathfrak{g})^H$-equivariant and the second isomorphism is $\mathcal{U}(\mathfrak{h}')^{G'}$-equivariant.

Proof. It is clear from following calculation,

$$\begin{aligned}
&\mathrm{Hom}_{\mathfrak{h}, \widetilde{K}_H}(\Theta(\rho), V_\tau) \\
\cong&\mathrm{Hom}_{(\mathfrak{h}, \widetilde{K}_H) \times (\mathfrak{g}', \widetilde{K}_{G'})}(\Theta(\rho) \otimes V_\rho, V_\tau \otimes V_\rho) \\
&\quad \text{(by Schur's lemma)} \\
\cong&\mathrm{Hom}_{(\mathfrak{h}, \widetilde{K}_H) \times (\mathfrak{g}', \widetilde{K}_{G'})}(\mathscr{Y} / \mathcal{N}_{\mathscr{Y}, V_\rho}, V_\tau \otimes V_\rho) \\
\cong&\mathrm{Hom}_{(\mathfrak{h}, \widetilde{K}_H) \times (\mathfrak{g}', \widetilde{K}_{G'})}(\mathscr{Y}, V_\tau \otimes V_\rho) \\
\cong&\mathrm{Hom}_{\mathfrak{g}', \widetilde{K}_{G'}}(\Theta(\tau), V_\rho).
\end{aligned} \tag{3.5}$$

□

Above proof is formal and lemma is actually true for any "see-saw pairs" of mutual commuting subgroups setting in a bigger group when certain from of Schur's lemma valide .

[1] By choosing an unitary character of $\mathbb{R}$.

Recall following Proposition.

Proposition 44 (Proposition 6, Chapter 2). *Let (G, G') and (H, H') be a see-saw pair in $\mathrm{Sp}(W)$ such that $H < G$ and $G' < H'$. Let ω be an oscillator representation of $\widetilde{\mathrm{Sp}}(W)$, then as a subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathscr{Y})$,*

$$\omega(\mathcal{U}(\mathfrak{g})^{H_{\mathbb{C}}}) = \omega(\mathcal{U}(\mathfrak{g})^{H}) = \omega(\mathcal{U}(\mathfrak{h}')^{G'}) = \omega(\mathcal{U}(\mathfrak{h}')^{G'_{\mathbb{C}}}). \tag{3.6}$$

Moreover, there exist a map $\Xi \colon \mathcal{U}(\mathfrak{g})^{H_{\mathbb{C}}} \to \mathcal{U}(\mathfrak{h}')^{G'_{\mathbb{C}}}$ (independent of real forms, may not unique and not be algebra homomorphism) such that $\omega(x) = \omega(\Xi(x))$.

Observe that there is an joint action $\mathcal{U}(\mathfrak{g})^H$ and $\mathcal{U}(\mathfrak{h}')^{G'}$ on

$$\mathrm{Hom}_{(\mathfrak{h}, \widetilde{K}_H) \times (\mathfrak{g}', \widetilde{K}_{G'})}(\mathscr{Y}, V_\tau \otimes V_\rho).$$

In the case of theta correspondence over $\mathbb{R}$, Proposition 44 will implie that $\mathcal{U}(\mathfrak{g})^H$ and $\mathcal{U}(\mathfrak{h}')^{G'}$ actions on the two sides of equation (3.4) determine each other.

3.3 Line bundles on symmetric spaces and Theta lifts of characters

In this section, we first present a theorem of Helgason [Hel64], describing the space of invariant differential operators on symmetric spaces. Helgason's original version [Hel64] and Shimura's extension [Shi90, Theorem 2.4] treat the case of connected semisimple Lie group G with H a maximal compact subgroup in G. See [Shi90], [Wal92] or [Zhu03] for reference. We can reform their results and obtain following slightly generalized version, Lemma 45, where G could be non-connected and reductive[2], and H could be non-compact. We also would like to point out that the right hand side of (3.7) is the space of G-invariant differential operators on a twisted line bundle.

Following lemma was also proved appeared in Section 8 of [Lep77]. But for completeness, we also give a proof in Appendix 3.A.

Lemma 45. *Let G be a real reductive group such that all simple factors of $\mathfrak{g}$ are classical Lie algebras. Let H be a symmetric subgroup of G in the sense that there is an involution σ on $\mathfrak{g}_0$ such that H is the subgroup of G with Lie algebra $\mathfrak{h}_0 = \mathfrak{g}_0^\sigma$ and meet all the connected component of G.*

[2] We adapt Wallach's definition of real reductive group (c.f. [Wal88, Section 2.1]).

Let $\mathcal{Z}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^G$ be the G-invariant subalgebra[3] *in $\mathcal{U}(\mathfrak{g})$. For an one-dimensional representation ρ of H, let $\mathcal{J}_\rho = \mathrm{Ann}_{\mathcal{U}(\mathfrak{h})}(\rho)$ be the annihilator ideal of ρ in $\mathcal{U}(\mathfrak{h})$. Then the natural homomorphism of ring*

$$\mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})^H/(\mathcal{J}_\rho\mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})^H) \tag{3.7}$$

is surjective.

Following lemma is immediate from above lemma (c.f. [Zhu03, Theorem 3.2]).

Lemma 46. *Let G and H be in the setting of Lemma 45, V be a $(\mathfrak{g}, K)$-module and ρ be a character of $(\mathfrak{h}, K_H)$. Then the $\mathcal{U}(\mathfrak{g})^H$ actions on $\mathrm{Hom}_{\mathfrak{h},K_H}(V, V_\rho)$ (or Ω_{V,V_ρ}) is determined by the $\mathcal{Z}(\mathfrak{g})$ action on V and the annihilator ideal $\mathcal{J}_\rho = \mathrm{Ann}_{\mathcal{U}(\mathfrak{h})}(\rho)$ of ρ.*

Proof. Let $0 \neq T \in \mathrm{Hom}_{\mathfrak{h},K_H}(V, V_\rho)$. For any $x \in \mathcal{U}(\mathfrak{g})^H$, choose $z \in \mathcal{Z}(\mathfrak{g})$ such that $x - z = ju$ for some $j \in \mathcal{J}_\rho$ and $u \in \mathcal{U}(\mathfrak{g})$. Therefore, $T(x - z) = Tju = jTu = 0$, i.e. $Tx = Tz$. This finished the proof. □

Now we retain the notations in Section 3.2. Let (G, G') and (H, H') from a see-saw pairs such that H is a symmetric subgroup of G. Then G' is automatically a symmetric subgroup of H' by the classification of reductive dual pairs.

Theorem 47. *Let $\rho \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$ be a character and $\tau \in \mathscr{R}(\mathfrak{h}, \widetilde{K}_H; \mathscr{Y})$. Then the $\mathcal{U}(\mathfrak{g})^H$ action on $\mathrm{Hom}_{\mathfrak{h},\widetilde{K}_H}(\Theta(\rho), V_\tau)$ is a character determined by the character*

$$\chi_\tau \colon \mathcal{Z}(\mathfrak{h}) \to \mathrm{Hom}_{\mathfrak{h},\widetilde{K}_H}(\tau, \tau) = \mathbb{C}$$

and the annihilator ideal $\mathcal{J}_\rho = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(\rho)$ of ρ.

Note that χ_τ may be slightly weaker than infinitesimal character since $\mathcal{Z}(\mathfrak{g})$ may be a proper subalgebra of the center of $\mathcal{U}(\mathfrak{g})$ by remarks after Lemma 45.

Proof. By Proposition 44, for any $x \in \mathcal{U}(\mathfrak{g})^H$, there is a $x' \in \mathcal{U}(\mathfrak{h}')^{G'}$ such that $\omega(x) = \omega(x')$. Now choose $z' \in \mathcal{Z}(\mathfrak{h}')$ such that $x' - z' \in \mathcal{J}_\rho\mathcal{U}(\mathfrak{h}')$ by Lemma 45.

[3] Our definition of $\mathcal{Z}(\mathfrak{g})$ could be smaller than the center of $\mathcal{U}(\mathfrak{g})$, for example, $\mathcal{U}(\mathfrak{so}(2n, \mathbb{C}))^{\mathrm{O}(2n)}$, which is the only "unusual" situation we will see in the context of theta correspondence.

Then again by Proposition 44, we can find $z \in \mathcal{Z}(\mathfrak{h})$ such that $\omega(z) = \omega(z')$. For any $0 \neq T \in \mathrm{Hom}_{(\mathfrak{h},\widetilde{K}_H)\times(\mathfrak{g}',\widetilde{K}_{G'})}(\mathscr{Y}, V_\tau \otimes V_\rho)$, by Lemma 43,

$$T\circ\omega(x) = T\circ\omega(x') = T\circ\omega(x'-z') + T\circ\omega(z') = T\circ\omega(z) = \tau(z)\circ T = \chi_\tau(z)T.$$

The Lemma follows. □

Furthermore, by Proposition 44, the choice of elements in above proof could be made independent of real forms. This is a crucial to us.

3.4 Transfer of K-types and the proof of Theorem A

We retain the notations in Section 3.1 and recall some basic facts of derived functor modules (c.f. [BW00, Section I.8] or [Wal88, Chapter 6] and Section 2.4).

First note, for any $V \in \mathscr{C}(\mathfrak{g}, M)$, as K-module,

$$\mathrm{R}^j(\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M})V = \mathrm{R}^j(\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M})V.$$

Set $\Gamma^j = \mathrm{R}^j\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M} = \mathrm{R}^j(\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M})$.

Let $U \in \mathscr{C}(\mathfrak{k}, M)$. Suppose $x \in \mathrm{Hom}_{\mathfrak{k},M}(V, V)$ and $T \in \mathrm{Hom}_{\mathfrak{k},M}(V, U)$ such that $T \circ x = cT$ with some constant $c \in \mathbb{C}$. Then, for $\Gamma^j T \in \mathrm{Hom}_K(\Gamma^j V, \Gamma^j U)$, we have

$$\Gamma^j T \circ \Gamma^j x = \Gamma^j(T \circ x) = \Gamma^j cT = c\Gamma^j T.$$

On the other hand, $\mathfrak{g}$-module structure on $\Gamma^j V$ is defined functorially. For $x \in \mathcal{U}(\mathfrak{g})^K$, which gives a $(\mathfrak{k}, M)$-equivariant map $x\colon V \to V$, x action on $\Gamma^j V$ is given by

$$\Gamma^j x\colon \Gamma^j V \to \Gamma^j V. \tag{3.8}$$

We summarise above discussion as following lemma.

Lemma 48. *Let $U \in \mathscr{C}(\mathfrak{k}, M)$, $V \in \mathscr{C}(\mathfrak{g}, M)$, $T \in \mathrm{Hom}_{\mathfrak{k},M}(V, U)$ and $x \in \mathcal{U}(\mathfrak{g})^K$. Suppose that x act on T by c, i.e. $T \circ x = cT$ with some constant $c \in \mathbb{C}$. Then x act on $\Gamma^j T$ by c. In particular, $\Gamma^j V$ has infinitesimal character if V has infinitesimal character.*

Remark: Following the discussion in [WZ04], for any irreducible $(\mathfrak{g}, K_1)$-module, $\Gamma^j V$ has infinitesimal character since V has infinitesimal character. Therefore every $(\mathfrak{g}, K)$-module generated by a finite dimensional K-

invariant subspace is admissible, and so has finite length[4]. Let $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$ be the annihilator ideals of V in $\mathcal{U}(\mathfrak{g})$. Then $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\Gamma^j V) \supset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$ and $\Gamma^i V$ has Gelfand-Kirillov dimension less than or equal to that of V.

Let $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ be a complex dual pair in $\mathrm{Sp}(W_{\mathbb{C}})$ for some complex symplectic space $W_{\mathbb{C}}$. For $i = 1, 2$, let $\mathfrak{g}_i$ be real forms of $\mathfrak{g}$ with complex conjugation σ_i and Cartan involution θ_i and satisfies equation 3.1. Let G'_i be real forms of $G'_{\mathbb{C}}$ such that (G_i, G'_i) form reductive dual pairs and $\mathscr{Y}_i$ be corresponding Fock modules. Consider following composition of transfer of K-types and big theta lifting:

$$\begin{array}{ccccc} \mathscr{R}(\mathfrak{g}', \widetilde{K}'_1; \mathscr{Y}_1) & \xrightarrow{\Theta} & \mathscr{R}(\mathfrak{g}, \widetilde{K}_1; \mathscr{Y}_1) & \xrightarrow{\Gamma^j} & \mathscr{C}(\mathfrak{g}, \widetilde{K}_2) \\ \rho & \longmapsto & \Theta(\rho) & \longmapsto & \Gamma^j\Theta(\rho). \end{array}$$

One may hope $\Gamma^j(\Theta(\rho)) = \Theta(\Gamma'^j(\rho))$ for some map $\Gamma' \colon \mathscr{R}(\mathfrak{g}', \widetilde{K}'_1; \mathscr{Y}_1) \to \mathscr{R}(\mathfrak{g}', \widetilde{K}'_2; \mathscr{Y}_2)$. However, example 41 already shows that the situation is much more subtle than this. As the example suggested, $\Gamma^j\Theta(\rho)$ could be non-zero and reducible for several j. Moreover, the composition components of $\Gamma^j\Theta(\rho)$ could be theta lifts from different G'_2s.

Let ρ_1 and ρ_2 be characters of $\widetilde{G}_1$ and $\widetilde{G}_2$ respectively, τ_1 be an irreducible $(\mathfrak{k}_2, \widetilde{K}_1 \cap \widetilde{K}_2)$-module and τ_2 an irreducible $\widetilde{K}_2$-module satisfies conditions in Theorem A. Now we prove our main theorem.

Proof of Theorem A. Let V_{τ_1} be a $(\mathfrak{k}_2, \widetilde{K}_1 \cap \widetilde{K}_2)$-module of type τ_1. Let

$$0 \neq T \in \mathrm{Hom}_{\mathfrak{k}_2, \widetilde{K}_1 \cap \widetilde{K}_2}(\Theta(\rho_1), V_{\tau_2})$$

be a map in assumption (b). Since τ_2 occur in the image of T, we can fix an irreducible $\widetilde{K}_2$-submodule U_{τ_2} of type τ_2 in $\Gamma^j\Theta(\rho_1)$ such that $\Gamma^j T(U_{\tau_2}) \neq 0$. Let

$$U := \mathcal{U}(\mathfrak{g}) U_{\tau_2} \subset \Gamma^j\Theta(\rho_1)$$

be the admissible $(\mathfrak{g}, \widetilde{K}_2)$-submodule of $\Gamma^j V$ generated by U_{τ_2}. View $\Gamma^j T|_U$ as a non-zero element in $\mathrm{Hom}_{\widetilde{K}_2}(U, V_{\tau_2})$ by restriction and post-composite the projection form $\Gamma^j V_{\tau_1}$ to the irreducible $\widetilde{K}_2$-module $V_{\tau_2} := \Gamma^j T(U_{\tau_2})$.

Let H be the maximal subgroup in G_1 with Lie algebra $\mathfrak{k}_2 \cap \mathfrak{g}_1$, which is generated by $\exp(\mathfrak{k}_2 \cap \mathfrak{g}_1)$ and $K_1 \cap K_2$. Clearly, $\mathrm{Lie}(H)_{\mathbb{C}} = \mathfrak{k}_2$ by equation 3.1. In our setting, $\widetilde{K}_1 \cap \widetilde{K}_2$ meet all the connected component of $\widetilde{K}_1$, $\widetilde{K}_2$ and $\widetilde{G}$. Moreover the action of double covers (e.g. $\widetilde{G}$) on Lie algebras

[4]Note there is only finite many isomorphism classes of irreducible $(\mathfrak{g}, K)$-module having a fixed infinitesimal character.

(e.g. $\mathfrak{g}$) factor through the linear group (e.g. G). Therefore, as subalgebras in $\mathcal{U}(\mathfrak{sp})$,

$$\begin{aligned}\mathcal{U}(\mathfrak{g})^{\widetilde{H}} =& \mathcal{U}(\mathfrak{g})^{H} = \mathcal{U}(\mathfrak{g})^{K_2} = \mathcal{U}(\mathfrak{g})^{\widetilde{K}_2} \quad \text{and} \\ \mathcal{U}(\mathfrak{k}_2)^{\widetilde{H}} =& \mathcal{U}(\mathfrak{k}_2)^{H} = \mathcal{U}(\mathfrak{k}_2)^{K_2} = \mathcal{Z}(\mathfrak{k}_2).\end{aligned} \tag{3.9}$$

Since V_{τ_2} is an irreducible $\widetilde{K}_2$-submodule in $\Gamma^j V_{\tau_1}$, $\mathcal{Z}(\mathfrak{k}_2)$ act on V_{τ_2}, V_{τ_1} and $\Gamma^j V_{\tau_1}$ by same character (c.f. Lemma 48). By Theorem 47, $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2}$ act on $\mathrm{Hom}_{\mathfrak{k}_2,\widetilde{K}_1}(V, V_{\tau_1})$ and $\mathrm{Hom}_{\widetilde{K}_2}(\Theta(\rho_2), V_{\tau_2})$ by the same character. By Lemma 48, $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2}$ act on $\Gamma^j T|_U$ also by this character.

On the other hand, $\mathrm{Hom}_{\widetilde{K}_2}(U, V_{\tau_2})$ is naturally isomorphic to the dual of $\mathrm{Hom}_{\widetilde{K}_2}(U_{\tau_2}, U(\tau_2))$ as $\mathcal{U}(\mathfrak{g})^K$-module. The irreducible $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2}$-submodule $\mathbb{C}\Gamma^j T|_U \subset \mathrm{Hom}_{\widetilde{K}_2}(U, V_{\tau_2})$ in correspond to an $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2}$-invariant irreducible quotient in $\mathrm{Hom}_{\widetilde{K}_2}(U_{\tau_2}, U(\tau_2))$, with $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2}$ action same as on $\Theta(\rho_2)(\tau_2)$ (note that $\Theta(\rho_2)$ is multiplicity free, hence $\mathrm{Hom}_{\widetilde{K}_2}(V_{\tau_2}, \Theta(\rho_2))$ is already an irreducible $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2}$-module).

Hence U has an irreducible quotient contains U_{τ_2}. Moreover it is isomorphic to the irreducible subquotient of $\Theta(\rho_2)$ containing $\widetilde{K}_2$-type τ_2 by a one-one correspondence between irreducible $(\mathfrak{g}, K)$-modules and $\mathcal{U}(\mathfrak{g})^K$-structure on K-isotypic components (c.f. Theorem 2). □

3.5 Examples

We retain the notation in Section 3.4. In this section, we will apply Theorem A and give families of examples on transfer of K-types.

In all these examples, (G_i, G_i') are reductive dual pairs in stable range, ρ_1 is certain unitary characters of G_1' and $\theta(\rho_1) = \Theta(\rho_1)$ is an unitary representation of $(\mathfrak{g}, \widetilde{K}_1)$. Moreover, $\theta(\rho_1)$ is discrete decomposable in sense of Kobayashi [Kob98] and decomposed into direct sum of irreducible unitary $(\mathfrak{k}_2, \widetilde{K}_1 \cap \widetilde{K}_2)$-modules. Therefore by the discussions in Section 2.4 (also see [WZ04]), $\Gamma^j V$ has following decomposition

$$\Gamma^j V = \bigoplus_{W_i} \Gamma_{W_i} V, \tag{3.10}$$

where $\bigwedge^j \mathfrak{k}_2/\mathfrak{m} = \bigoplus W_i$ as direct sum of irreducible modules of M. We will show that $\Gamma_{W_i} V$ is either zero or a theta lift (so irreducible). Therefore equation 3.10 solved the possible reducibility of $\Gamma^j V$ in all our cases.

In section 3.5.1, we will study the transfer of singular unitary lowest weight modules which are theta lifts of unitary character. In Section 3.5.2, we will discuss another type of examples initiated by a joint work with Loke and Tang [LMT11b]. As corollaries of these examples provide some examples beyond derived functor modules of theta lifts of characters.

3.5.1 Transfer of unitary lowest weight modules lifted from unitary characters

In analogy to the cohomological induction, the derived functor modules of singular unitary lowest weight modules have been studied in [EW80] [Fra91] [Wal94] [WZ04].

To have unitary lowest weight modules, $\mathfrak{g}$ should be Hermitian symmetric. We will study three families of examples where $\mathfrak{g}$ has root systems of A, C, D respectively.

	$G := G_1$	H	G'_1	H'	$G'_{p,q}$	Sp
Type A	$\mathrm{U}(n,n)$	$\mathrm{U}(r,s)\times\mathrm{U}(s,r)$	$\mathrm{U}(m)$	$\mathrm{U}(m)\times\mathrm{U}(m)$	$\mathrm{U}(p,q)$	$\mathrm{Sp}(4nm,\mathbb{R})$
Type C	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{U}(r,s)$	$\mathrm{O}(m)$	$\mathrm{U}(m)$	$\mathrm{O}(p,q)$	$\mathrm{Sp}(2nm,\mathbb{R})$
Type D	$\mathrm{O}^*(2n)$	$\mathrm{U}(r,s)$	$\mathrm{Sp}(m)$	$\mathrm{U}(2m)$	$\mathrm{Sp}(p,q)$	$\mathrm{Sp}(4nm,\mathbb{R})$

Table 3.1: Transfer of unitary lowest weight modules

Follow notation in Table 3.1. Let σ be the involution of $G_1 = G$ such that $H := G_1^\sigma \cong \mathrm{U}(r,s)$. H' is the centralizer of H in Sp. Then (G_1, G'_1) and (H, H') form a see-saw pair in Sp. Fix a Cartan involution θ of Sp commute with σ such that $K_1 = \mathrm{U}\cap G_1$ is a maximal compact subgroup of G_1 and $G'_1 \subset \mathrm{U}$. Let $\mathscr{Y}$ be the Fock module of the oscillator representation ω of $\widetilde{\mathrm{Sp}}$ which is a $(\mathfrak{sp}, \widetilde{\mathrm{U}})$-module. Let $G_2 = G_{\mathbb{C}}^{\sigma\theta}$ be the set of $\sigma\theta$ fixed points in $G_{\mathbb{C}}$. Then $K_2 := G_2^\sigma$ is a maximal compact subgroup of G_2, $\mathfrak{k}_2 = \mathfrak{h}$ and $M := K_1 \cap K_2 = H^\theta$ is the maximal compact subgroup of H. Let

$$\Gamma^j = R^j\Gamma_{\mathfrak{g},\widetilde{M}}^{\mathfrak{g},\widetilde{K}_2} = R^j\Gamma_{\mathfrak{k}_2,\widetilde{M}}^{\mathfrak{k}_2,\widetilde{K}_2}.$$

By above construction, we have $G_2 \cong G_1 \cong G$. Let $\theta^{p,q}$ (resp. $\theta^{m,0}$) be the theta lifting from $G'_{p,q}$ (resp. $G'_1 \cong G'_{m,0}$) to G_2 (resp. G_1). We can summarise the main result in this section as following form. For precise statements in each cases, see Theorem 56, Theorem 58 and Theorem 60

Theorem. *Let (G_1, G'_1) be in stable range. Let ρ be an unitary character of G'_1 which is trivial restricted on $\mathfrak{g}'$.*

$$\Gamma^j\theta^{m,0}(\rho) = \bigoplus \theta^{p,q}(\rho_{p,q})$$

where p, q run over a subset of positive integers such that $p + q = m$ and $\rho_{p,q}$ are unitary characters of $G'_{p,q}$ determined by p, q and ρ.

Remark:

1. Since $G \cong G_1 \cong G_2$, the above theorem could be view as an construction of G-modules from lowest weight G-modules. If we consider the derived functor Γ^j in all degrees together , there is a simple formula

$$\bigoplus_{j \in \mathbb{N}} \Gamma^j \theta^{m,0}(\rho) \cong \bigoplus_{p+q=m} \theta^{p,q}(\rho_{p,q}).$$

2. When ρ corresponde to the trivial representation of G'_1, Frajria [Fra91] studied $\Gamma^j \theta(\rho)$ for its first non-zero degree. He proved such $\Gamma^j V$ is irreducible and unitarizable and expect these examples could be fit in the dual pair correspondence. Wallach and Zhu made a precise conjecture [WZ04, Conjecture 5.1] on the decomposition of $\Gamma^j \theta(\rho)$ for Type C.

3. As an exploration of general cases, we also studied the transfer of unitary lowest weight module lifted from arbitrary finite dimensional representations of G'_1 for Type C. However, currently, we do not known whether they are certain theta lifts.

Before go to case by case calculations, we supply a proof on the decomposition of $\theta(\rho)$. It is a consequence of the direct sum decomposition of $\mathscr{Y}$ under compact dual pairs (c.f. Theorem 9 in Section 2.3.5) and well known to experts.

Temporally drop the assumption that (G_1, G'_1) is in the stable range. Note that (G_1, G'_1) and (H, H') are compact dual pairs, let $\rho \in \mathcal{R}(\widetilde{G}'_1; \mathscr{Y})$ be a finite dimensional irreducible genuine representation of $\widetilde{G}'_1$, then $\Theta(\rho)$ is an unitary lowest weight $(\mathfrak{g}, \widetilde{K}_1)$-module.

Lemma 49. *Under $(\mathfrak{h}, \widetilde{M})$-action, $\Theta(\rho)$ decompose into direct sum of irreducible unitary lowest modules:*

$$\Theta(\rho) = \bigoplus_{\tau^{\mu}_{\widetilde{H}'} \in \mathcal{R}(\widetilde{H}'; \mathscr{Y})} n_\mu L_{\widetilde{H}}(\mu).$$

Here $n_\mu = \dim \operatorname{Hom}_{\widetilde{G}'}(\rho, \tau^{\mu}_{\widetilde{H}'}) = \dim \operatorname{Hom}_{\widetilde{G}'}(\tau^{\mu}_{\widetilde{H}'}, \rho)$ is the multiplicity of $L_{\widetilde{H}}(\mu)$ occur in $\Theta(\rho)$. In particular, $n_\mu = 0$ if $L_{\widetilde{H}}(\mu)$ dose not occur in the decomposition of $\Theta(\rho)$.

Proof. Since $\Theta(\rho) \otimes \rho$ can be identify with the ρ-isotypic subspace of $\mathscr{Y}$ which is the image of a $\widetilde{G}'$-equivariant projection, the $\widetilde{H}'$-invariant decomposition $\mathscr{Y} = \bigoplus_{\mu\in\mathcal{R}(\widetilde{H}';\mathscr{Y})} L_{\widetilde{H}}(\mu)\otimes\tau^{\mu}_{\widetilde{H}'}$ induce the decomposition of

$$\Theta(\rho)\otimes\rho = \mathscr{Y}(\rho) = \bigoplus_{\tau^{\mu}_{\widetilde{H}'}\in\mathcal{R}(\widetilde{H}';\mathscr{Y})} L_{\widetilde{H}}(\mu)\otimes \mathrm{Hom}_{\widetilde{G}'}(\rho,\tau^{\mu}_{\widetilde{H}'})\otimes\rho. \tag{3.11}$$

Projection to a particular vector of ρ, for example the highest weight vector, (3.11) implies the desired decomposition of $\Theta(\rho)$. This proved the Lemma. □

As a corollary of Lemma 49, we have.

Lemma 50. *As $\widetilde{K}_2$-module,*

$$\Gamma^j\Theta(\rho) = \bigoplus_{\tau^{\mu}_{\widetilde{H}'}\in\mathcal{R}(\widetilde{H}';\mathscr{Y})} n_\mu \Gamma^j L_{\widetilde{H}}(\mu).$$

Now we begin the case by case calculation.

3.5.1.1 Type C

The main object is to prove example 41. We retain the notation in Section 3.4. Now $(G_1, G_1') = (\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(m))$ and $(H, H') = (\mathrm{U}(r,s), \mathrm{U}(m))$ with $r+s=m$. $G_2 \cong \mathrm{Sp}(2n,\mathbb{R})$, $K_2 \cong \mathrm{U}(n)$ and $M \cong \mathrm{U}(r)\times\mathrm{U}(s)$, $G_2' = \mathrm{O}(p,q)$ such that $m = p+q$, $r\geq p$, $s\geq q$. So (G_i, G_i') are all in the stable range. Let $\theta^{p,q}$ be the theta lifting form $\mathrm{O}(p,q)$ to $\mathrm{Sp}(2n,\mathbb{R})$. We will only assume ρ is an irreducible finite dimensional representation of G_1' at first.

Now let

$$\mu := (a_1,\cdots,a_p,-b_q,\cdots,-b_1) = (\mathbf{a}, -\mathbf{b}^r),$$

and $\{\, a_i \,\}$, $\{\, b_i \,\}$ are non-increasing non-negative integers. Recall the description of $\Gamma^j L_{\widetilde{\mathrm{U}}(r,s)}(\mu)$.

Lemma 51 (Lemma 25, Section 2.4.3)**.** *Let* $\mathbf{a}' := \mathbf{a} - (s-q)\mathbf{1}_p$, $\mathbf{b}' := \mathbf{b} - (r-p)\mathbf{1}_q$ *and*

$$\Gamma(\mu) = (\mathbf{a}', \mathbf{0}, -\mathbf{b}'^r).$$

Then $\Gamma^j L_{\widetilde{\mathrm{U}}(r,s)}(\mu) \neq 0$ *only if* $\mathbf{a}'\geq 0$ $\mathbf{b}'\geq 0$, *i.e.* $a_p \geq s-q$, $b_q \geq r-p$. *In* *this case,*

$$\Gamma^j L_{\widetilde{\mathrm{U}}(n)}(\mu) = \begin{cases} \Gamma_{W_{p,q}} L_{\widetilde{\mathrm{U}}(n)}(\mu) = \tau_{\widetilde{\mathrm{U}}(n)}^{\Gamma(\mu)+\frac{p-q}{2}} & \text{if } j = rs-(r-p)(s-q) \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

where $W_{p,q} \subset \bigwedge^j \mathfrak{h}/\mathfrak{m}$ *is the unique irreducible* $\mathrm{U}(r)\times\mathrm{U}(s)$*-module in* $\bigwedge \mathfrak{h}/\mathfrak{m}$ *with isomorphic to*

$$\tau_{\mathrm{U}(r)}^{((s-q)\mathbf{1}_p,\mathbf{0})} \otimes \tau_{\mathrm{U}(s)}^{(\mathbf{0},-(r-p)\mathbf{1}_q)}.$$

For p, q such that $r \geq p$, $s \geq q$, let

$$\mathcal{M}_{p,q}^0 = \left\{ \mu = (a_1, \cdots, a_p, -b_q, \cdots, -b_1) \;\middle|\; \begin{array}{l} a_1 \geq \cdots \geq a_p \geq 0, \\ b_1 \geq \cdots \geq b_q \geq 0 \end{array} \right\},$$
$$\mathcal{M}_{p,q} = \left\{ \mu = (a_1, \cdots, a_p, -b_q, \cdots, -b_1) \;\middle|\; \begin{array}{l} a_1 \geq \cdots \geq a_p \geq s-q, \\ b_1 \geq \cdots \geq b_q \geq r-p \end{array} \right\}.$$

Now $\mathcal{M}_{p,q}^0$ form a collection of disjoint subsets of the heights weight of $\mathfrak{gl}(m,\mathbb{C})$ such that

$$\Theta(\rho) = \bigoplus_{\substack{r\geq p \\ s\geq q}} \bigoplus_{\mu\in\mathcal{M}_{p,q}^0} \mathrm{Hom}_{\widetilde{\mathrm{O}}(m)}(\rho, \tau_{\widetilde{\mathrm{U}}(m)}^{\mu+\frac{r-s}{2}\mathbf{1}_m}) \otimes L_{\widetilde{\mathrm{U}}(r,s)}(\mu). \tag{3.13}$$

For $\mu \in \mathcal{M}_{p,q}^0$, $\Gamma_{W_{p,q}} L(\mu) \neq 0$ if and only if $\mu \in \mathcal{M}_{p,q}$.

Definition 52. We call a set $\mathscr{X}$ of $(\mathfrak{g}, K)$-modules has *disjoint K-spectrums* if any $\sigma \in \hat{K}$ only occur in at most one $(\mathfrak{g}, K)$-module in $\mathscr{X}$.

Remark: A example of $\mathscr{X}$ is the set of irreducible subquotients of degenerate principle series.

Proposition 53. *Fix* $\rho \in \mathscr{R}(G_1'; \mathscr{Y})$.

(i) *As* $(\mathfrak{g}, \widetilde{K}_2)$*-module*

$$\Gamma^j\Theta(\rho) = \bigoplus_{\substack{W\subset\bigwedge^j\mathfrak{p} \\ W \text{ irreducible}}} \Gamma_W\Theta(\rho) = \bigoplus_{j=rs-(r-p)(s-q)} \Gamma_{W_{p,q}}\Theta(\rho).$$

(ii) *As* $\widetilde{\mathrm{U}}(n)$*-module,*

$$\Gamma_{W_{p,q}}\Theta(\rho) = \bigoplus_{\mu\in\mathcal{M}_{p,q}} n_\mu \tau_{\widetilde{\mathrm{U}}(n)}^{\Gamma(\mu)+\frac{p-q}{2}}.$$

where $n_\mu = \dim \mathrm{Hom}_{\widetilde{\mathrm{O}}(m)}(\rho, \tau_{\widetilde{\mathrm{U}}(m)}^{\mu+\frac{r-s}{2}\mathbf{1}_m})$.

(iii) $\Gamma_{W_{p,q}}\Theta(\rho)$ *is non-zero and of finite length.*

(iv) The set of subquotients of $\Gamma_{W_{p,q}}\Theta(\rho)$ *for all* p, q *have disjoint* K*-types.*

Proof. Part (i) and (ii) are clear from Lemma 51.

To prove (iii), we need following theorem on the stability of branching from Sato [Sat94].

Theorem 54. *Let* Λ^+ *be the lattice of highest weight of* $\mathrm{U}(m)$ *such that, for* $\lambda \in \Lambda^+$, $\lambda \geq 0$. *Let* $\Lambda^+(\mathrm{O}(m))$ *be the subset of* Λ^+ *such that* $\tau^{\mu}_{\mathrm{U}(m)}$ *has* $\mathrm{O}(m)$ *invariant, which is the subset of* Λ^+ *with even entries :*

$$\Lambda^+(\mathrm{O}(m)) = \left\{ \lambda \in \Lambda^+ \;\middle|\; \lambda \in (2\mathbb{N})^m \right\}.$$

For any $\lambda \in \Lambda^+$ *and* $\rho \in \widehat{\mathrm{O}(m)}$, *let* $m(\lambda, \rho) = \dim \mathrm{Hom}_{\mathrm{O}(m)}(\tau^{\lambda}_{\mathrm{U}(m)}, \rho)$. *Then, there exists* $\eta^M \in \Lambda^+(\mathrm{O}(m))$ *such that*

$$m(\lambda + \eta^M, \rho) = m(\lambda + \eta^M + \eta, \rho), \quad \forall\, \eta \in \Lambda^+(\mathrm{O}(m)).$$

The integer $m(\lambda+\eta^M, \rho)$ *does not depend on the choice of* η^M. *Denote the integer by* $m([\lambda], \rho)$ *and call it the* stable branching coefficient. *Moreover,* $m([\lambda], \rho) \geq m(\lambda, \rho)$.

To show $\Gamma_{W_{p,q}}V$ is nonzero. We only have to show there is a $\mu \in M_{p,q}$ such that $m(\mu, \rho) \neq 0$. Note that there always exist a μ, such that $m(\mu, \rho) \geq 1$ by the construction of irreducible $\mathrm{O}(m)$-module (c.f. [GW09]). Now $m(\mu+2\gamma, \rho) \geq m(\mu, \rho) \geq 1$ for $\gamma \gg 0$ sufficiently large. Choose γ such that $(\mu+2\gamma)[p]-(\mu+2\gamma)[p+1] \gg 0$. Notice that $m(\mu+2k\mathbf{1}_m, \rho) = m(\mu, \rho)$ for any $k \in \mathbb{Z}$, since it just twist $\tau^{\mu}_{\mathrm{U}(m)}$ by $\det^2$ (trivial when restrict on $\mathrm{O}(m)$). Therefore $\mu' = \mu + 2\gamma - 2\lfloor\frac{(\mu+2\gamma)_p+(\mu+2\gamma)_{p+1}}{4}\rfloor\mathbf{1}_m$ will be an element in $\mathcal{M}_{p,q}$ such that $m(\mu', \rho) \geq 1$. This proves first part of (iii). We will show $\Gamma_{W_{p,q}}$ is finite length in the proof of (iv).

Now we prove (iv). Note that, as $\mathcal{U}(\mathfrak{g})^{\widetilde{H}} \times (\mathfrak{h}, \widetilde{M})$-module,

$$\Theta(\rho) = \bigoplus \mathrm{Hom}_{\mathfrak{h},\widetilde{M}}(L_{\widetilde{H}}(\mu), \Theta(\rho)) \otimes L_{\widetilde{H}}(\mu). \tag{3.14}$$

By Lemma 51, as $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2} \times \widetilde{K}_2$-module, (note that $\mathcal{U}(\mathfrak{g})^{\widetilde{K}_2} = \mathcal{U}(\mathfrak{g})^{\widetilde{H}}$)

$$\Gamma_{W_{p,q}}\Theta(\rho) = \bigoplus_{\mu \in \mathcal{M}_{p,q}} \mathrm{Hom}_{\mathfrak{h},\widetilde{M}}(L_{\widetilde{H}}(\mu), \Theta(\rho)) \otimes \tau^{\Gamma(\mu)+\frac{p-q}{2}}_{\widetilde{\mathrm{U}}(n)}$$

Recall Lemma 43, as $\omega(\mathcal{U}(\mathfrak{g})^H) = \omega(\mathcal{U}(\mathfrak{h}')^{G_1'})$ module,

$$\mathrm{Hom}_{\mathfrak{h},\widetilde{M}}(\Theta(\rho), L_{\widetilde{H}}(\mu)) \cong \mathrm{Hom}_{\widetilde{G}_1'}(\tau^{\mu}_{\widetilde{H}'}, \rho).$$

Since $\widetilde{\mathrm{O}}(m)$ is the maximal compact subgroup of $\widetilde{\mathrm{GL}}(m, \mathbb{R})$, by viewing $\tau^{\mu}_{\widetilde{H}'}$ as an irreducible $\widetilde{\mathrm{GL}}(m, \mathbb{R})$-module, we conclude that $\mathcal{U}(\mathfrak{h}')^{G'}$ act on $\mathrm{Hom}_{\widetilde{G}_1'}(\tau^{\mu}_{\widetilde{H}'}, \rho)$ irreducibly. Hence $\mathcal{U}(\mathfrak{g})^{\widetilde{H}}$ act on $\mathrm{Hom}_{\mathfrak{h},\widetilde{M}}(L_{\widetilde{H}}(\mu), \Theta(\rho))$ irreducibly.

On the other hand, non-zero $\Gamma^* L_{\widetilde{H}}(\mu)$ are different for different μ since they have different central character and infinitesimal characters. Hence the $\widetilde{K}_2$-isotypic components in $\Gamma^*\Theta(\rho)$ has irreducible $(\widetilde{K}_2, \mathcal{U}(\mathfrak{g})^{\widetilde{K}_2})$-action. This implies the disjointness of K-spectrum.

Note that all sub-quotients in $\Gamma^j\Theta(\rho)$ have the same infinitesimal character and central character. Therefore $\Gamma^j\Theta(\rho)$ has finite length, since it is K_2-admissible. In particular $\Gamma_{W_{p,q}}\Theta(\rho)$ has finite length. □

Let $\mathbf{1}^{\xi,\eta}$ be the character of $\mathrm{O}(p, q)$ which is $\det^{\xi} \otimes \det^{\eta}$ when restricted on $\mathrm{O}(p) \times \mathrm{O}(q)$. These provide all of the four characters of $\mathrm{O}(p, q)$ represented by $\mathbf{1}^{\xi,\eta}$ with $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$ when p, q are both non-zero (i.e. $\mathrm{O}(p, q)$ is non-compact). When one of p, q is zero, $\mathrm{O}(p, q)$ is compact and it only has two characters: trivial and the determinate. By twisting genuine characters, we identify $\mathrm{O}(p, q)$-module with $\widetilde{\mathrm{O}}(p, q)$ such that following description of theta lifting of $\mathbf{1}^{\xi,\eta}$ holds.

Theorem 55 ([KR90], [Zhu92], [HZ02, Theorem 2.3] or Section 2.3.6)**.** *Suppose $p + q \leq n$. Let $\theta^{p,q}(\mathbf{1}^{\xi,\eta})$ be the theta lifting of $\mathbf{1}^{\xi,\eta}$. Then the $\widetilde{\mathrm{U}}(n)$-types in $\theta^{p,q}(\mathbf{1}^{\xi,\eta})$ are exactly of following form:*

$$\tau_{\widetilde{\mathrm{U}}(n)}^{(a_1,\cdots,a_q,0,\cdots,0,-b_q,\cdots,-b_1)+\frac{p-q}{2}}$$

where $a_1 \geq \cdots \geq a_p \geq 0$, $b_1 \geq \cdots \geq b_q \geq 0$, $a_j \equiv \xi \pmod 2$ and $b_j \equiv \eta \pmod 2$.

Remark: By Li's work [Li89], $\theta^{p,q}(\mathbf{1}^{\xi,\eta})$ are all different irreducible unitary representations of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$. In particular $\theta^{p,q}(\mathbf{1}^{\xi,\eta})$ and $\theta^{q,p}(\mathbf{1}^{\xi,\eta})$ are different although $\mathrm{O}(p, q) \cong \mathrm{O}(q, p)$. This difference come from the two different choices of unitary characters of $\mathbb{R}$ when we define oscillator representation.

Theorem 56. *Let $\Gamma_{p,q} := \Gamma_{W_{p,q}}$, we have*

$$\Gamma_{p,q}\theta^{m,0}(\mathbf{1}^{\epsilon,\epsilon}) \cong \theta^{p,q}(\mathbf{1}^{\xi,\eta})$$

where $\xi = \epsilon - (s-q) \pmod 2$ *and* $\eta = \epsilon - (r-p) \pmod 2$. *Moreover,*

$$\Gamma^j \theta^{m,0}(\mathbf{1}^{\epsilon,\epsilon}) = \bigoplus_{\substack{j=rs-(r-p)(s-q),\\ p\le r,\, q\le s}} \Gamma_{p,q}\theta^{m,0}(\mathbf{1}^{\epsilon,\epsilon}) \cong \bigoplus_{\substack{j=rs-(r-p)(s-q),\\ p\le r,\, q\le s}} \theta^{p,q}(\mathbf{1}^{\xi,\eta}).$$

□

Proof. Let $G_2' = \mathrm{O}(p,q)$. Note that all modules are $\widetilde{K}_2$-multiplicity free. Combine Theorem A and the description of $\widetilde{K}_2$-spectrum of $\theta^{p,q}(\mathbf{1}^{\xi,\eta})$ finished the proof. □

Remark: Here $\theta^{m,0}(\mathbf{1}^{0,0})$ is the dual of $L(-\frac{m}{2}\Lambda_1)$ in [WZ04]. By changing the of positive root system to the negative one, we can see above theorem confirm the Conjecture 5.1 in [WZ04]. Theorem 56 implies that $\Gamma_{p,q}\theta^{m,0}(\mathbf{1}^{\epsilon,\epsilon})$ is irreducible and unitarizable since $\theta^{p,q}(\mathbf{1}^{\xi,\eta})$ is by Li's work [Li89] of theta lifting in stable range. While there are some criterions for the irreducibility and unitarity of derived functor modules [EPWW85][Fra91][Wal94][WZ04], our approach is conceptually simpler. Moreover, these criterions seems not applicable to the lifts of determinate since the original lowest weight modules do not have scaler K-type.

3.5.1.2 Type D

We consider Type D root system first since it is simpler. Now $(G_1, G_1') = (\mathrm{O}^*(2n), \mathrm{Sp}(m))$ and $n \geq 2m$. $H \cong \mathrm{U}(r,s)$ with $r+s=n$. $(G_2, G_2') = (\mathrm{O}^*(2n), \mathrm{Sp}(p,q))$ with $p+q=m$. Since $\widetilde{\mathrm{Sp}}(p,q) \cong \mathbb{Z}/2\mathbb{Z} \times \mathrm{Sp}(p,q)$, there is only one genuine character $\mathbf{1}$, which is trivial when restrict on $\mathfrak{sp}(p,q)$. Moreover $\widetilde{\mathrm{O}}^*(2n) \cong \mathbb{Z}/2\mathbb{Z} \times \mathrm{O}^*(2n)$, we identify the genuine representations of $\widetilde{G}_j$ with G_j by twist with the unique genuine character (or equivalently, by restrict on the identity component of $\widetilde{\mathrm{O}}^*(2n)$). We use similar convention for $\widetilde{\mathrm{Sp}}(p,q)$-module. Let

$$\Gamma^j = R^j\Gamma^{\mathfrak{so}(2n,\mathbb{C}),\mathrm{U}(n)}_{\mathfrak{so}(2n,\mathbb{C}),\mathrm{U}(r)\times\mathrm{U}(s)}.$$

Let $\theta^{p,q}$ be the theta lifting from $\mathrm{Sp}(p,q)$ to $\mathrm{O}^*(2n)$. Following lemma on the K-spectrum implies Theorem 58 by the same argument in previous section.

Lemma 57. *(i) The* $\mathrm{U}(n)$*-type ocurre in* $\theta^{p,q}(\mathbf{1})$ *has exactly the form* $\tau^{\mu}_{\mathrm{U}(n)}$, *where*

$$\mu = (a_1, a_1, a_2, a_2, \cdots, a_p, a_p, \mathbf{0}_{n-2(p+q)}, -b_q, -b_q, \cdots, -b_2, -b_2, -b_1, -b_1) + (p-q)\mathbf{1}_n$$

and $a_1 \geq \cdots a_p \geq 0,\ b_1 \geq \cdots b_q \geq 0.$

(ii) As $(\mathfrak{gl}(n,\mathbb{C}), \mathrm{U}(r)\times\mathrm{U}(s))$*-module,*

$$\theta^{m,0}(\mathbf{1}) = \bigoplus_{\substack{2p\leq r\\ 2q\leq s}} \bigoplus_{\mu\in\mathcal{M}^0_{p,q}} L_{\mathrm{U}(r,s)}(\mu)$$

where

$$\mathcal{M}^0_{p,q} = \left\{ \mu = \begin{array}{c} (a_1,a_1,a_2,a_2,\cdots a_p,a_p,\\ -b_q,-b_q,\cdots,-b_1,-b_1) \end{array} \middle| \begin{array}{l} a_1\geq\cdots\geq a_p\geq 0,\\ b_1\geq\cdots\geq b_q\geq 0 \end{array} \right\}$$

and $L_{\mathrm{U}(r,s)}(\mu)$ *is with respect to dual pair* $(\mathrm{U}(r,s),\mathrm{U}(2m))$.

Theorem 58. *Let* $\Gamma_{p,q} = \Gamma_{W_{2p,2q}}$*. Then*

$$\Gamma_{p,q}\theta^{m,0}(\mathbf{1}) \cong \theta^{p,q}(\mathbf{1})$$

and

$$\Gamma^j\theta^{m,0}(\mathbf{1}) = \bigoplus_{\substack{j=rs-(r-2p)(s-2q),\\ 2p\leq r,\,2q\leq s}} \Gamma_{p,q}\theta^{m,0}(\mathbf{1}) \cong \bigoplus_{\substack{j=rs-(r-2p)(s-2q),\\ 2p\leq r,\,2q\leq s}} \theta^{p,q}(\mathbf{1}).$$

Proof. By Lemma 57 and same argument as previous section. □

3.5.1.3 Type A

Let $(G_1,G_1') = (\mathrm{U}(n_1',n_2'),\mathrm{U}(m))$ such that $n_1',n_2'\geq m$. Set $H = \mathrm{U}(r_1,s_1)\times \mathrm{U}(s_2,r_2)$ such that $r_1+s_2 = n_1'$ and $s_1+r_2 = n_2'$. Let $n_1 = r_1+s_1$, $n_2 = r_2+s_2$. and $(G_2,G_2') = (\mathrm{U}(n_1,n_2),\mathrm{U}(p,q))$ with $p+q = m$. Also assume stable range condition $n_1,n_2\geq m$. We will transfer $\widetilde{\mathrm{U}}(n_1',n_2')$ representation into $\widetilde{\mathrm{U}}(n_1,n_2)$ representation. By identify genuine character of $\widetilde{\mathrm{U}}(p,q)$ with its restriction on Lie algebra, for integer or half-integer α, let $\det^\alpha$ be the genuine character of $\widetilde{\mathrm{U}}(p,q)$ which restrict on $\mathfrak{gl}(p+q,\mathbb{C})$ has (highest) weight $\alpha\mathbf{1}_{p+q}$. Let $\theta^{p,q}_{n_1,n_2}$ be the theta lifting of from $\mathrm{U}(p,q)$ to $\mathrm{U}(n_1,n_2)$.

There are two sources of complexity for pair $(\mathrm{U}(n_1,n_2),\mathrm{U}(m))$: (i) n_1 and n_2 could be different; (ii) there are infinite many unitary characters of $\mathrm{U}(m)$. In fact, the set of unitary characters of $\mathrm{U}(m)$ is isomorphic to $\mathbb{Z}$ (the Pontryagin dual of $\mathrm{U}(1)$). The theta lifts varies in a tricky way when n_1-n_2 and unitary character changes(c.f. [PT02]). Here we only consider a very special unitary character.

Lemma 59. *(i) The* $\widetilde{\mathrm{U}}(n_1)\times\widetilde{\mathrm{U}}(n_2)$*-type occur in* $\theta^{p,q}_{n_1,n_2}(\det^\alpha)$ *are of form* $\tau^{\mu_1}_{\widetilde{\mathrm{U}}(n_1)}\otimes\tau^{\mu_2}_{\widetilde{\mathrm{U}}(n_2)}$ *where*

$$\mu_1 =(\mathbf{a},\mathbf{0},-\mathbf{b}^t)+\frac{p-q}{2}\mathbf{1}_{n_1}\qquad \mu_2 =(\mathbf{c},\mathbf{0},-\mathbf{d}^t)+\frac{q-p}{2}\mathbf{1}_{n_2}$$

and $\mathbf{a},\mathbf{d}$ *(resp.* $\mathbf{b},\mathbf{c}$*) are arrays of decreasing non-negative integers of length* p *(resp.* q*) such that*

$$\begin{aligned}\mathbf{c} =&\frac{n_1-n_2}{2}+\mathbf{b}+\alpha\mathbf{1}_q\\ \mathbf{d} =&\frac{n_1-n_2}{2}+\mathbf{a}-\alpha\mathbf{1}_p.\end{aligned}$$

(ii) Suppose $\alpha=\frac{n_1'-n_2'}{2}$*, then* $\theta^{m,0}_{n_1',n_2'}(\det^\alpha)$ *is a lowest weight module with scaler* K*-type. As* $\mathfrak{u}(r_1,s_1)\times\mathfrak{u}(s_2,r_2)$*-module*

$$\theta^{m,0}(\det^\alpha)=\bigoplus_{p,q}\bigoplus_{\mu_1,\mu_2}L_{\widetilde{\mathrm{U}}(r_1,s_1)}(\mu_1)\otimes L_{\widetilde{\mathrm{U}}(s_2,r_2)}(\mu_2)$$

where

$$\mu_1 =(\mathbf{a},-\mathbf{b}^t)\qquad\qquad \mu_2 =(\mathbf{c},-\mathbf{d}^t)$$

such that $\mathbf{a}$ *(resp.* $\mathbf{b}$*) is array of decreasing non-negative integers of length* p *(resp.* q*) and* $\mathbf{a}=\mathbf{d}$ *and* $\mathbf{b}=\mathbf{c}$*. Here* p,q *run over non-negative integers such that* $p+q=m$*,* $p\le r_1,r_2$ *and* $q\le s_1,s_2$*. Such* p,q *exists by the "stable range" assumption, i.e.* $n_1,n_2,n_1',n_2'\ge m$*.*

Theorem 60. *Let* $\alpha=\frac{n_1'-n_2'}{2}$ *and* $\Gamma_{p,q}=\Gamma_{W_{p,q}\otimes W_{q,p}}$*, then*

$$\Gamma_{p,q}\theta^{m,0}_{n_1',n_2'}(\det^\alpha)\cong\theta^{p,q}_{n_1,n_2}(\det^\alpha).$$

and

$$\Gamma^j\theta^{m,0}_{n_1',n_2'}(\det^\alpha)=\bigoplus_{\substack{p\le r_1,r_2\\ q\le s_1,s_2}}\Gamma_{p,q}\theta^{m,0}_{n_1',n_2'}(\det^\alpha)\cong\bigoplus_{\substack{p\le r_1,r_2\\ q\le s_1,s_2}}\theta^{p,q}_{n_1,n_2}(\det^\alpha).$$

In particular, let $n_1=n_2=n_1'=n_2'=n$*,* $r_1=r_2=r$*,* $s_1=s_2=s$ *and* $\theta^{p,q}(\mathbf{1}):=\theta^{p,q}_{n,n}(\det^0)$*. Then*

$$\Gamma^j\theta^{m,0}(\mathbf{1})=\bigoplus_{\substack{j=2(rs-(r-p)(s-q)),\\ p\le r,q\le s}}\Gamma_{p,q}\theta^{m,0}(\mathbf{1})\cong\bigoplus_{\substack{j=2(rs-(r-p)(s-q)),\\ p\le r,q\le s}}\theta^{p,q}(\mathbf{1})$$

3.5.2 Transfer of theta lifts of unitary characters and unitary lowest weight module of Hermitian symmetric groups

In this section we first study the transfer of theta lifts of characters from non-compact groups. As an application and an extension of these results, we discuss the transfer of unitary lowest weight modules via a restriction method from [LMT11b].

3.5.2.1 Notation

We will consider series of reductive groups in Table 3.2, where $(G^{p,q}, G')$ form a reductive dual pair.

	$G^{p,q}$	G'	Stable range	j_0
Case $\mathbb{R}$	$\mathrm{O}(p,q)$	$\mathrm{Sp}(2n,\mathbb{R})$	$p,q \geq 2n$,$\max\{\,p,q\,\} > 2n$	nr
Case $\mathbb{C}$	$\mathrm{U}(p,q)$	$\mathrm{U}(n_1,n_2)$	$p,q \geq n_1+n_2$	$(n_1+n_2)r$
Case $\mathbb{H}$	$\mathrm{Sp}(p,q)$	$\mathrm{O}^*(2n)$	$p,q \geq n$	$2nr$

Table 3.2: List of dual pairs I

First recall that the double covering of G' is depends on the parity of $p+q$. We use the restriction on $\mathfrak{g}'$ to parametrize genuine $(\mathfrak{g}', \widetilde{K}')$-module since G' is always connected in our case.

Fix integers r, r' such that $r + r' = q$. Define

$$L_{\widetilde{G}'}(\mu) := \theta(\tau^{\mu}_{\widetilde{G}^{r',0}}) \tag{3.15}$$

to be the theta lift of $\tau^{\mu}_{\widetilde{G}^{r',0}}$ from $\widetilde{G}^{r',0}$ to $\widetilde{G}'$. Here we use letter L to emphasis that $L_{\widetilde{G}'}(\mu)$ is a lowest weight module.

Warning: Although $G^{r'} := G^{r',0} \cong G^{0,r'}$, the theta lifting maps for pairs $(G^{r',0}, G')$ and $(G^{0,r'}, G')$ are different. The two theta lifting maps are dual to each other (see [Prz88, Theorem 5.5]): $L_{\widetilde{G}'}(\mu)$ is the theta lift of $\tau^{\mu}_{\widetilde{G}^{r'}}$ from $G^{r'}$ to G' if and only if the contragredient $\left(L_{\widetilde{G}'}(\mu)\right)^*$ is the theta lift of $(\tau^{\mu}_{\widetilde{G}^{r'}})^*$ from $G^{r'}$ to G'. In case $\mathbb{R}$ and $\mathbb{H}$, $\widetilde{G}^{r'}$-module is self-dual i.e. $(\tau^{\mu}_{\widetilde{G}^{r'}})^* = \tau^{\mu}_{\widetilde{G}^{r'}}$.

Let $\theta^{p,q}$ (resp. $\Theta^{p,q}$) be the theta lifting (resp. full theta lifting map) from G' to $G^{p,q}$. Consider a symmetric subgroup H of $G^{p,q}$ in the form of $H_1 \times H_2 \cong G^{p,r} \times G^{0,r'}$. Let $\mathfrak{h} := \mathfrak{h}_1 \oplus \mathfrak{h}_2 := \mathrm{Lie}(H_1)_{\mathbb{C}} \oplus \mathrm{Lie}(H_2)_{\mathbb{C}}$. Assume the corresponding involution defining H commute with the fixed Cartan involution for $G^{p,q}$. Let $K_{H_1} = G^{p,0} \times G^{0,r}$ be the corresponding maximal compact subgroup of H_1. Meanwhile, $H_2 = G^{0,r'}$ is already compact.

Let $\mathscr{Y}$ be the Fock model for pair $(G^{p,q}, G')$. When restricted on the pair $(H_1 \times H_2, G' \times G')$,

$$\mathscr{Y} = \mathscr{Y}_1 \otimes \mathscr{Y}_2$$

where $\mathscr{Y}_j$ is the Fock model of reductive dual pair (H_j, G') with $j = 1, 2$.

Fix a character $\rho \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$. $(L_{G'}(\mu) \otimes \rho)|_{\mathfrak{g}'}$ determines a genuine $\widetilde{G}'$-module, where $\widetilde{G}'$ is the double covering of G' defined by pair $(G^{p,r}, G')$.

Definition 61. We define the $(\mathfrak{h}_1, \widetilde{G}^{p,0} \times \widetilde{G}^{0,r})$-module

$$\Theta^{p,r}_{\rho}(\mu) := \begin{cases} \Theta^{p,r}(L_{\widetilde{G}'}(\mu) \otimes \rho) & \text{if } L_{\widetilde{G}'}(\mu) \otimes \rho \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y}_1); \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

We define $\theta^{p,r}_{\rho}(\mu)$ in a same way, so $\theta^{p,r}_{\rho}(\mu)$ is the unique irreducible quotient of $\Theta^{p,r}_{\rho}(\mu)$ if it is non-zero.

When $\rho|_{\mathfrak{g}'}$ is trivial, $\theta^{p,r}_{\rho}(\mu)$ is the two step theta lift of $\tau^{\mu}_{\widetilde{G}^{r',0}}$ if it is non-zero. In this case we omit the subscript ρ and so we have $\Theta^{p,r}(\mu) = \Theta^{p,r}(L_{\widetilde{G}'}(\mu))$ and $\theta^{p,r}(\mu) = \theta^{p,r}(L_{\widetilde{G}'}(\mu))$.

3.5.2.2 Restriction method.

The following lemma is from Loke.

Lemma 62 (Lemma 4.2.1 [LL06]). *Let ρ be a character in $\mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$. Then,*

(i) As $(\mathfrak{h}_1, \widetilde{K}_{H_1}) \times \widetilde{H}_2$-modules,

$$\Theta^{p,q}(\rho) = \bigoplus_{(\tau^{\mu}_{\widetilde{H}_2})^* \in \mathscr{R}(\widetilde{H}_2; \mathscr{Y}_2)} \Theta^{p,r}_{\rho}(\mu) \otimes (\tau^{\mu}_{\widetilde{H}_2})^*, \tag{3.17}$$

(ii) $\tau^{\mu}_{\widetilde{H}_2} \leftrightarrow \Theta^{p,r}_{\rho}(\mu)$ is an one-one correspondence and there is a correspondence of their infinitesimal characters, which is independent of real forms.

When ρ is an unitary character and $(G^{p,q}, G')$ is in the stable range,

(iii) $\Theta^{p,r}_{\rho}(\mu) = \theta^{p,r}_{\rho}(\mu)$ is irreducible and unitarizable if it is non-zero.

Proof. Since $\widetilde{H}_2$ is compact, $\Theta^{p,q}(\rho)$ can be decompose into $\widetilde{H}_2$-isotypic component, i.e.

$$\Theta^{p,q}(\rho) = \bigoplus_{\tau \in \widehat{\widetilde{H}_2}} A(\tau) \otimes \tau,$$

where $A(\tau) \cong \operatorname{Hom}_{\widetilde{H}_2}(\tau, \Theta^{p,q}(\rho))$. The projection into $\widetilde{H}_2$-isotypic component τ will give rise a non-zero map from $\mathscr{Y}_2$ to τ, hence τ runs over $\mathscr{R}(\widetilde{H}_2; \mathscr{Y}_2)$. Now we show that $A((\tau^{\mu}_{\widetilde{H}_2})^*) \cong \Theta^{p,r}_{\rho}(\mu)$. Consider commutative diagram (3.18). All arrows are $(\mathfrak{h}_1, \widetilde{K}_{H_1}) \times \widetilde{H}_2 \times (\mathfrak{g}', \widetilde{K}')$ equivariant.

$$\begin{array}{ccccc}
\mathscr{Y}_1 \otimes \mathscr{Y}_2 & \twoheadrightarrow & \Theta^{p,q}(\rho) \otimes \rho & \twoheadrightarrow & A(\tau^{\mu}_{\widetilde{H}_2}) \otimes (\tau^{\mu}_{\widetilde{H}_2})^* \otimes \rho \\
\downarrow \cong & & & \nearrow \eta \quad \searrow & \uparrow \bar{\eta} \\
\mathscr{Y}_1 \otimes \bigoplus_{(\tau^{\mu}_{\widetilde{H}_2})^* \in \mathscr{R}(\widetilde{H}_2; \mathscr{Y}_2)} L^*_{\widetilde{G}'}(\mu) \otimes (\tau^{\mu}_{\widetilde{H}_2})^* & \twoheadrightarrow & \mathscr{Y}_1 \otimes L^*_{\widetilde{G}'}(\mu) \otimes (\tau^{\mu}_{\widetilde{H}_2})^* & \twoheadrightarrow & \Theta^{p,r}_{\rho}(\mu) \otimes \rho \otimes (\tau^{\mu}_{\widetilde{H}_2})^*
\end{array} \tag{3.18}$$

Map η factor through $\bar{\eta}$ by the definition of maximal Howe quotient $\Theta^{p,r}_{\rho}(\mu)$ and following natural isomorphism[5]

$$\operatorname{Hom}_{\mathfrak{g}', \widetilde{K}'}(\mathscr{Y}_1 \otimes L^*_{\widetilde{G}'}(\mu), \rho) \cong \operatorname{Hom}_{\mathfrak{g}', \widetilde{K}'}(\mathscr{Y}_1, L_{\widetilde{G}'}(\mu) \otimes \rho). \tag{3.19}$$

On the other hand,

$$\bigoplus_{(\tau^{\mu}_{\widetilde{H}_2})^* \in \mathscr{R}(\widetilde{H}_2; \mathscr{Y}_2)} \Theta^{p,r}_{\rho}(\mu) \otimes (\tau^{\mu}_{\widetilde{H}_2})^* \otimes \rho$$

is a quotient of $\mathscr{Y}$ such that $\mathfrak{g}'$ act by character ρ and $\Theta^{p,q}(\rho)$ is the maximal one. So there is a natural surjection from left hand side of (3.17) to its right hand side. This implies $A((\tau^{\mu}_{\widetilde{H}_2})^*) \cong \Theta^{p,r}_{\rho}(\mu)$.

For (ii), the correspondence is one-one since theta lifting map is an bijection. The correspondence of infinitesimal characters is from the infinitesimal character correspondence of theta lifting [Prz96].

Now let ρ be a unitary character and $(G^{p,q}, G')$ be in the stable range. $\Theta^{p,q}(\rho) = \theta^{p,q}(\rho)$ is an unitary representation (c.f. Section 2.3.6). Clearly $\Theta^{p,r}_{\rho}(\mu)$ is unitarizable with unitary structure inherent from $\theta^{p,q}(\rho)$. Therefore it is a direct sum of its irreducible component, since it has finite length. On the other hand, Howe's theory of theta lifting says $\Theta^{p,r}_{\rho}(\mu)$ has only one irreducible quotient. This implies that $\Theta^{p,r}_{\rho}(\mu)$ is irreducible and isomorphic to $\theta^{p,r}_{\rho}(\mu)$. This complete the proof of the lemma. □

The following lemma is easy consequence.

[5]In general, $\operatorname{Hom}_{\mathfrak{g},K}(V \otimes U, W) \cong \operatorname{Hom}_{\mathfrak{g},K}(V, \operatorname{Hom}_{\mathbb{C}}(U, W)_{K-\text{finite}})$. In our case, $\operatorname{Hom}_{\mathbb{C}}(U, \rho)_{K-\text{finite}} \cong U^* \otimes \rho$ since ρ is a character.

Lemma 63. *As $\widetilde{G}^{p,0} \times \widetilde{G}^{0,q}$ module $\Theta^{p,q}(\rho)$,*

$$\Theta^{p,q}(\rho) = \bigoplus_{\nu \in \Lambda} \tau^{\xi(\nu)}_{\widetilde{G}^{p,0}} \otimes \tau^{\nu}_{\widetilde{G}^{0,q}}$$

where Λ is a set of array of (half-)integers parametrizing $\widetilde{G}^{0,q}$-modules and ξ maps ν into another array of (half-)integers correspond to $\widetilde{G}^{p,0}$-module.

As $\widetilde{G}^{p,0} \times \widetilde{G}^{0,r}$-module,

$$\Theta^{p,r}_{\rho}(\mu) = \bigoplus_{\nu \in \Lambda, \tau^{\nu'}_{\widetilde{G}^{0,r}} \in \widehat{\widetilde{G}^{0,r}}} \tau^{\xi(\nu)}_{\widetilde{G}^{p,0}} \otimes \tau^{\nu'}_{\widetilde{G}^{0,r}} \cdot \dim \mathrm{Hom}_{\widetilde{G}^{0,r} \times \widetilde{G}^{0,r'}} (\tau^{\nu'}_{\widetilde{G}^{0,r}} \otimes (\tau^{\mu}_{\widetilde{G}^{0,r'}})^*, \tau^{\nu}_{\widetilde{G}^{0,q}}),$$

where the dimension of the Hom*-space is the multiplicity of K-type.* □

Remark: It is difficult to calculate the explicit K-type formula for $\Theta^{p,r}_{\rho}(\mu)$ in general, since it is equivalent to the branching problem of finite dimensional representations. However, one can show the occurrence of some special K-type by above lemma. Combining with its unitarizability and informations on infinitesimal characters, this is enough for us to show that $\theta^{p,r}_{\rho}(\mu)$ is certain $A_{\mathfrak{q}}(\lambda)$ with non-zero cohomology by [VZ84].

For now on, let ρ be an unitary character in $\mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$ and (G, G') is in the stable range. Let $G_1 = G^{p,q}$, $G'_1 = G'_2 = G'$. Construct G_2 as in Section 3.5.1 such that $G_2 \cong G^{p+r,q-r}$, its maximal compact subgroup $K_2 \cong G^{p+r,0} \times G^{0,q-r}$ and $\mathrm{Lie}(K_2)_{\mathbb{C}} = \mathfrak{h}$. Now $M := H \cap K_2 = K_{H_1} \times H_2$ is the maximal compact subgroup of H. Recall that

$$\Gamma^j = R^j(\Gamma^{\mathfrak{g}, \widetilde{K}_2}_{\mathfrak{g}, \widetilde{M}}) \circ \mathcal{F}^{\mathfrak{g}, \widetilde{M}}_{\mathfrak{g}, \widetilde{K}_1} : \mathscr{C}(\mathfrak{g}, \widetilde{K}_1) \to \mathscr{C}(\mathfrak{g}, \widetilde{K}_2). \tag{3.20}$$

Let $\Gamma^j_c \triangleq R^j(\Gamma^{\mathfrak{h}_1, \widetilde{G}^{p+r,0}}_{\mathfrak{h}_1, \widetilde{K}_{H_1}})$. Then

$$\begin{aligned} R^j(\Gamma^{\mathfrak{h}, \widetilde{K}_2}_{\mathfrak{h}, \widetilde{M}})(\theta^{p,r}_{\rho}(\mu) \otimes (\tau^{\mu}_{\widetilde{H}_2})^*) =& R^j(\Gamma^{\mathfrak{h}_1 \oplus \mathfrak{h}_2, \widetilde{G}^{p+r,0} \times \widetilde{H}_2}_{\mathfrak{h}_1 \oplus \mathfrak{h}_2, \widetilde{K}_{H_1} \times \widetilde{H}_2})(\theta^{p,r}_{\rho}(\mu) \otimes (\tau^{\mu}_{\widetilde{H}_2})^*) \\ \cong& \Gamma^j_c(\theta^{p,r}_{\rho}(\mu)) \otimes (\tau^{\mu}_{\widetilde{H}_2})^* \end{aligned}$$

by the definition of Zuckerman functor and the fact that $\widetilde{H}_2$ is already compact. Therefore, by Lemma 62, as $\widetilde{K}_2$-module,

$$\Gamma^j \theta^{p,q}(\rho) = \bigoplus_{\tau^{\mu}_{\widetilde{H}_2} \in \mathscr{R}(\widetilde{H}_2; \mathscr{Y}_2)} \Gamma^j_c(\theta^{p,r}_{\rho}(\mu)) \otimes (\tau^{\mu}_{\widetilde{H}_2})^*.$$

Since $\theta^{p,r}_{\rho}(\mu)$ is unitarizable, $\Gamma^j_c(\theta^{p,r}_{\rho}(\mu))$ is non-zero, only if $\theta^{p,r}_{\rho}(\mu)$ is certain $A_{\mathfrak{q}}(\lambda)$ and its infinitesimal character should be non-singular by [VZ84].

3.5.2.3 Statement of the main theorem

For simplicity[6], we will only consider the case that $\rho|_{\mathfrak{g}'}$ is trivial. This condition implies that $p+q$ will be even in Case $\mathbb{R}$ and Case $\mathbb{C}$. We also only stay in the stable range, hence $\theta^{p,q}(\mathbf{1})$ is non-zero and unitary. Recall Definition 61, we have $\theta^{p,r}(\mu)$.

Now we state the main theorem in this section.

Theorem 64. *Assume $p+q$ is even in case $\mathbb{R}$ and case $\mathbb{C}$. Let $\theta^{p,q}(\mathbf{1})$ be the theta lifts of trivial representation of G'.*

(i) If $(G^{p+r,q-r}, G')$ is outside the stable range, then $\Gamma^j\theta^{p,q}(\mathbf{1}) = 0$.

(ii) If $(G^{p+r,q-r}, G')$ is in the stable range, then

$$\Gamma^j\theta^{p,q}(\mathbf{1}) = \begin{cases} \theta^{p+r,q-r}(\mathbf{1}) & \text{if } j = j_0 \\ 0 & j < j_0, \end{cases}$$

where $j_0 = nr, (n_1+n_2)r$ and $2nr$ in case $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ respectively as listed in Table 3.2.

Proof. By Theorem A and K-spectrum comparison, the theorem follows. □

Remark:

1. We will provide the K-spectrum data case by case in the end of this section.

2. For all cases, $\widetilde{G}' \cong \mathbb{Z}/2\mathbb{Z} \times G'$. Let ς' be the unique genuine character of $\widetilde{G}'$ trivial on G'. We identify the genuine $\widetilde{G}'$-module with the G'-module $\varsigma' \otimes V$, i.e. $\theta^{p,q}(\mathbf{1})$ means $\theta^{p,q}(\varsigma')$.

3. The infinitesimal character of $\theta^{p,r}(\mu)$ and therefore $\Gamma_c^j(\theta^{p,r}(\mu))$ is determined by μ. So the K_c-module $\Gamma_c^j(\theta^{p,r}(\mu))$ is determined[7] by μ if it is non-zero. On the other hand, the infinitesimal character correspondence between $\mathfrak{h}_1$ and $\mathfrak{h}_2$-module is independent of real form. Therefore, if the derived functor module is non-zero, the K-type satisfied the correct constrain, so it is reasonable to expect the transfer of $\theta^{p,r}(\mathbf{1})$ is some copies of $\theta^{p+r,q-r}(\mathbf{1})$. Careful computations are needed to determine the multiplicity for each degree j.

[6]Trivial representation is the only unitary character of $\mathrm{Sp}(2n,\mathbb{R})$(or $\mathrm{O}^*(2n,\mathbb{R})$). However, powers of determinate form a finite family of unitary character of $\mathcal{U}(n_1, n_2)$.

[7]For case $\mathbb{R}$, up to determinate character

4. We will identify $\theta^{p,r}(\mu)$ as $A_{\mathfrak{q}}(\lambda)$ when its infinitesimal characters is non-singular. The θ-stable parabolic $\mathfrak{q}$ is independent of μ and λ. Now $\Gamma^j\theta^{p,r}(\mu)$ is m_j copies of K-module with highest weight λ where

$$m_j = \dim \mathrm{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\bigwedge^{j-j_0}(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C})$$

and $j_0 = \dim \mathfrak{u}\cap\mathfrak{p}$ (c.f. [VZ84] or Appendix 2.4). Hence, in fact, $\Gamma^j\theta^{p,q}(\mathbf{1})$ will be m_j copies of $\theta^{p+r,q-r}(\mathbf{1})$. We only state the result for the first non-zero degree j_0, since derived functor modules of higher degree do not construct any new objects.

5. The result of case $\mathbb{R}$ may consider as a generalization of an example of Enright et al. [EPWW85, Section 8]. They studied the transfer of ladder representation of $\mathrm{SO}(2,2n)$, which is theta lift of the trivial representation from $\mathrm{SL}(2,\mathbb{R}) = \mathrm{Sp}(2,\mathbb{R})$.

By a restriction argument, we obtain a result on the transfer of theta lifts of lowest weight modules, see Corollary 65.

Corollary 65. *Let Γ^j be the derived functor transfers $\widetilde{G}^{p,r}$-module to $\widetilde{G}^{p+t,r-t}$-module as in*[8] *(3.20). Suppose $(G^{p,r+r'}, G')$ is in the stable range such that $p+r+r'$ is even in Case $\mathbb{R}$ and $\mathbb{C}$. Then, for $0 < t \le r$,*

(i) if $(G^{p+t,r+r'-t}, G')$ is not in the stable range, $\Gamma^j\theta^{p,r}(\mu) = 0$;

(ii) if $(G^{p+t,r+r'-t}, G')$ is in the stable range,

$$\Gamma^j\theta^{p,r}(\mu) = \begin{cases} \theta^{p+t,r-t}(\mu) & \text{if } j = j_0 \\ 0 & \text{otherwise,} \end{cases}$$

where $j_0 = nt, (n_1+n_2)t$ and $2nt$ in case $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ respectively.

Proof. The proof is exactly same as in [LMT11b]. For completeness we repeat the proof here. By Lemma 62,

$$\theta^{p,r}(\mu) = \mathrm{Hom}_{\widetilde{G}^{0,r'}}((\tau^{\mu}_{\widetilde{G}^{0,r'}})^*, \theta^{p,r+r'}(\mathbf{1})),$$

where the $\mathfrak{g}^{p,r}$-module structure is from $\theta^{p,r+r'}(\mathbf{1})$. Therefore

$$\begin{aligned}\Gamma^j\theta^{p,r}(\mu) \cong& \mathrm{Hom}_{\widetilde{G}^{0,r'}}((\tau^{\mu}_{\widetilde{G}^{0,r'}})^*, \Gamma^j\theta^{p,r+r'}(\mathbf{1})) \\ =& \mathrm{Hom}_{\widetilde{G}^{0,r'}}((\tau^{\mu}_{\widetilde{G}^{0,r'}})^*, \theta^{p+t,r+r'-t}(\mathbf{1})) = \theta^{p+t,r-t}(\mu).\end{aligned}$$

[8] Replace q, r by r, t respectively.

Above equation is an $(Lie(G^{p+t,r-t})_{\mathbb{C}}, \widetilde{G}^{p+t,r-t})$-module isomorphism, because its action commute with $\widetilde{G}^{0,r'}$. □

Now we provide data for the proof of Theorem 64.

3.5.2.4 Case $\mathbb{R}$

This case is treated in [LMT11b].

First we write the double cover of $\mathrm{O}(p,q)$ explicitly [9]. We have

$$\widetilde{\mathrm{O}}(p+q,\mathbb{C}) \cong \left\{ (g,\varepsilon) \in \widetilde{\mathrm{O}}(p+q,\mathbb{C}) \times \mathbb{C}^\times \,\middle|\, \det(g) = \varepsilon^n \right\}.$$

Define $\varsigma\colon \widetilde{\mathrm{O}}(p+q,\mathbb{C}) \to \mathbb{C}^\times$ by $(g,\varepsilon) \mapsto \varepsilon$, which is a genuine character of $\widetilde{\mathrm{O}}(p+q,\mathbb{C})$ and its restriction on $\widetilde{\mathrm{O}}(p,q)$ is also a genuine character.

We identify the set of genuine representation $\widetilde{O}(p,q)$ with the set of representations of $\mathrm{O}(p,q)$ by map $\pi \mapsto \varsigma \otimes \pi$. Since ς is actually a genuine character on $\widetilde{O}(p+q,\mathbb{C})$, we have:

$$\varsigma \otimes \Gamma^j V = \Gamma^j(\varsigma \otimes V),$$

and both side factor through corresponding linear group. Therefore, we reduce the transfer problem to the linear group.

Note that $p+q$ is even, $\widetilde{\mathrm{Sp}}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$ is a trivial central extension of $\mathrm{Sp}(2n,\mathbb{R})$. Let ς' be the unique genuine character of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Again we will identify the genuine representation π' of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ with the representation $\pi' \otimes \varsigma'$ of $\mathrm{Sp}(2n,\mathbb{R})$. Now consider the transfer of the theta lifts $\theta^{p,q}(\mathbf{1})$.

Lemma 66. *(i) As $\mathrm{O}(p) \times \mathrm{O}(q)$-module,*

$$\theta^{p,q}(\mathbf{1}) = \bigoplus_{\mathbf{a}=(a_1,\cdots,a_n)} \det{}^p \tau_{\mathrm{O}(p)}^{(\mathbf{a}-\frac{p-q}{2}\mathbf{1}_n,\mathbf{0}_{p-n})} \otimes \tau_{\mathrm{O}(q)}^{(\mathbf{a},\mathbf{0}_{q-n})}, \tag{3.21}$$

[9] One may also use co-cycle $c(g,h) = (\det(g),\det(h))_{\mathbb{R}}^n$ to define the double cover, where$(\ ,\ )_{\mathbb{R}}$ is the Hilbert symbole of $\mathbb{R}$, i.e. $(a,b)_{\mathbb{R}}$ is 1 if a or b positive, is -1 otherwise. Then $\widetilde{\mathrm{O}}(p+q,\mathbb{C}) = \mathrm{O}(p+q,\mathbb{C}) \times \mu_2$ ($\mu_2 = \{\pm 1\}$) with group law $(g_1,\epsilon_1)(g_2,\epsilon_2) = (g_1g_2, c(g_1,g_2)\epsilon_1\epsilon_2)$. Define $\widetilde{\mathrm{O}}(p,q)$ to be the inverse image of $\mathrm{O}(p,q) \subset \mathrm{O}(p+q,\mathbb{C})$. Define

$$\chi_0(g,\epsilon) = \begin{cases} \epsilon & \text{if } \det(g) > 0, \\ -\epsilon i & \text{if } \det(g) < 0. \end{cases} \qquad \varsigma(g,\epsilon) = \begin{cases} \epsilon & \text{if } n \text{ is even;} \\ \chi_0(g,\epsilon)^{-1} & \text{if } n \text{ is odd.} \end{cases}$$

In particular, the double covering of $\mathrm{O}(p,q)$ and generic character ς only depends on the dimension of the symplectic space.

where the direct sum is run over $\mathbf{a} = (a_1, \cdots, a_n)$ *such that* a_i *are non-negative integers and* $a_1 \geq \cdots \geq a_n \geq \min\left\{ \frac{p-q}{2}, 0 \right\}$.

(ii) Restrict on $(\mathfrak{so}(p+r), \mathrm{O}(p) \times \mathrm{O}(r)) \times \mathrm{O}(r')$,

$$\theta^{p,q}(\mathbf{1}) = \bigoplus_{\mu \in \widehat{\mathrm{O}(r')}} \theta^{p,r}(\mu) \otimes \tau^{\mu}_{\mathrm{O}(r')}$$

where $\mu = (\mu_1, \cdots, \mu_n)$, $\theta^{p,r}(\mu)$ *is irreducible unitarizable or zero depending on whether* $\tau^{\mu}_{\mathrm{O}(r')}$ *occur in some* $\tau^{(\mathbf{a}, \mathbf{0}_{q-n})}_{\mathrm{O}(q)}$ *of* (3.21).

(iii) Let $l = \frac{p+r-r'}{2}$ *and*

$$\xi(\mu) = (\mu - l\mathbf{1}_n, \mathbf{0}).$$

Then $\theta^{p,r}(\mu)$ *have regular infinitesimal character if and only if* $r' \geq 2n$ *and* $\mu_n \geq l$.

(iv) When $\theta^{p,r}(\mu)$ *has regular infinitesimal character, as* $\mathfrak{so}(p+r, \mathbb{C})$*-module,*

$$\theta^{p,r}(\mu) = \begin{cases} A_{\mathfrak{q}}(\xi(\mu)) & \text{if } p > 2n, \\ A_{\mathfrak{q}}(\xi(\mu)) \oplus A_{\mathfrak{q}'}(\xi'(\mu)) & \text{if } p = 2n, \end{cases}$$

where $A_{\mathfrak{q}'}(\xi'(\mu))$ *is the conjugate of* $A_{\mathfrak{q}}(\xi(\mu))$ *by an element in* $\mathrm{O}(2n) \setminus \mathrm{SO}(2n)$.

(v) Under the assumption of (iv), if W *is an irreducible* $\mathrm{O}(p) \times \mathrm{O}(r)$*-submodule of* $\bigwedge^j \mathfrak{p}$ *with highest weight* $(l\mathbf{1}_n, \mathbf{0}_{p-n})$,

$$\Gamma_W(\theta^{p,r}(\mu)) = \det{}^n \tau^{\xi(\mu)}_{\mathrm{O}(p+r)}.$$

In fact, such module fist occur in $\bigwedge^{nr} \mathfrak{p}$ *and occur in* $\bigwedge^{nr+j} \mathfrak{p}$ *with multiplicity* $m_{nr+j} = \dim(\mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{k}}(\bigwedge^j \mathfrak{l} \cap \mathfrak{p}, \mathbb{C}))$ *(c.f.* [VZ84]*).*

3.5.2.5 Case $\mathbb{H}$

In this case, both $\widetilde{\mathrm{Sp}}(p,q)$ and $\widetilde{O}^*(2n)$ are trivial coverings. Hence, there is only one genuine character of both group. We also reduce to linear groups as in Section 3.5.2.4.

Lemma 67. *(i) As* $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$*-module,*

$$\theta^{p,q}(\mathbf{1}) = \bigoplus_{\mathbf{a} = (a_1, \cdots, a_n)} \tau^{(\mathbf{a} - (p-q)\mathbf{1}_n, \mathbf{0}_{p-n})}_{\mathrm{Sp}(p)} \otimes \tau^{(\mathbf{a}, \mathbf{0}_{q-n})}_{\mathrm{Sp}(q)}. \tag{3.22}$$

(ii) *Restrict on* $(\mathfrak{sp}(p+r), \mathrm{Sp}(p)\times\mathrm{Sp}(r))\times\mathrm{Sp}(r')$,

$$\theta^{p,q}(\mathbf{1}) = \bigoplus_{\mu\in\widehat{\mathrm{Sp}(r')}} \theta^{p,r}(\mu)\otimes\tau^{\mu}_{\mathrm{O}(r')}$$

where $\mu=(\mu_1,\cdots,\mu_n)$, $\theta^{p,r}(\mu)$ *is irreducible unitarizable or zero depending on whether* $\tau^{\mu}_{\mathrm{Sp}(r')}$ *occur in some* $\tau^{(\mathbf{a},\mathbf{0}_{q-n})}_{\mathrm{Sp}(q)}$.

(iii) *Let* $l=p+r-r'$ *and*

$$\xi(\mu)=(\mu-l\mathbf{1}_n,\mathbf{0}).$$

Then $\theta^{p,r}(\mu)$ *have regular infinitesimal character if and only if* $r'\geq n$ *and* $\mu_n\geq l$.

(iv) *When* $\theta^{p,r}(\mu)$ *has regular infinitesimal character, as* $\mathfrak{sp}(p+r,\mathbb{C})$*-module,*

$$\theta^{p,r}(\mu)=A_{\mathfrak{q}}(\xi(\mu)).$$

(v) *Under the assumption of (iv), if* W *is an irreducible* $\mathrm{Sp}(p)\times\mathrm{Sp}(r)$*-submodule of* $\bigwedge^j\mathfrak{p}$ *with highest weight* $(2r\mathbf{1}_n,\mathbf{0}_{p-n})$,

$$\Gamma_W(\theta^{p,r}(\mu))=\tau^{\xi(\mu)}_{\mathrm{Sp}(p+r)}.$$

In fact, such module fist occur in $\bigwedge^{2nr}\mathfrak{p}$ *and occur in* $\bigwedge^{2nr+j}\mathfrak{p}$ *with multiplicity* $m_{2nr+j}=\dim(\mathrm{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\bigwedge^j\mathfrak{l}\cap\mathfrak{p},\mathbb{C}))$ *(c.f.* [VZ84]*).*

3.5.2.6 Case $\mathbb{C}$

Note that $p+q$ is even and $\widetilde{\mathrm{U}}(n_1,n_2)\cong\mathbb{Z}/2\mathbb{Z}\times\mathrm{U}(n_1,n_2)$.

Lemma 68. (i) *As* $\widetilde{\mathrm{U}}(p)\times\widetilde{\mathrm{U}}(q)$*-module*

$$\theta^{n_1,n_2}(\mathbf{1})=\bigoplus_{\mathbf{a},\mathbf{b}}\tau^{(\mathbf{a}-\frac{p-q}{2},\mathbf{0},-(\mathbf{b}^t-\frac{p-q}{2}))+\frac{n_1-n_2}{2}}_{\widetilde{\mathrm{U}}(p)}\otimes\tau^{(\mathbf{b},\mathbf{0},-\mathbf{a}^t)+\frac{n_2-n_1}{2}}_{\widetilde{\mathrm{U}}(q)} \tag{3.23}$$

where $\mathbf{a}$ *(resp.* $\mathbf{b}$*) are arrays of decreasing non-negative integers of length* l *(resp.* n*) such that* $\mathbf{b}-\frac{p-q}{2}$ *and* $\mathbf{a}-\frac{p-q}{2}$ *are also non-negative.*

(ii) *Restrict on* $(\mathfrak{u}(p+r),\widetilde{\mathrm{U}}(p)\times\widetilde{\mathrm{U}}(r))\times\widetilde{\mathrm{U}}(r')$,

$$\theta^{p,q}(\mathbf{1})=\bigoplus_{\mu\in\widehat{\widetilde{\mathrm{U}}(r')}}\theta^{p,r}(\mu)\otimes(\tau^{\mu}_{\widetilde{\mathrm{U}}(r')})^*$$

where $\theta^{p,r}(\mu)$ *is irreducible unitarizable or zero depending on whether* $(\tau^{\mu}_{\widetilde{\mathrm{U}}(r')})^*$ *occur in some* $\tau^{(\mathbf{b},\mathbf{0},-\mathbf{a}^t)+\frac{n_2-n_1}{2}}_{\widetilde{\mathrm{U}}(q)}$ *appeared in* (3.23).

(iii) Let $l = \frac{p+r-r'}{2}$*.For* $\mu = (\mathbf{a},\mathbf{0},-\mathbf{b}^t) + \frac{n_1-n_2}{2}$*, let*

$$\xi(\mu) = (\mathbf{a} - l\mathbf{1}_{n_1}, \mathbf{0}_{p-(n_1+n_2)}, -(\mathbf{b}^t - l\mathbf{1}_{n_2}), \mathbf{0}_r) + \frac{n_1-n_2}{2}.$$

Then $\theta^{p,r}(\mu)$ *have regular infinitesimal character if and only if* $r' \geq n_1+n_2$*,* $a_{n_1} \geq l$ *and* $b_{n_2} \geq l$*.*

(iv) When $\theta^{p,r}(\mu)$ *has regular infinitesimal character, as* $\mathfrak{u}(p+r,\mathbb{C})$*-module,*

$$\theta^{p,r}(\mu) = A_{\mathfrak{q}}(\xi(\mu)).$$

(v) Under the assumption of (iv), let W *be a irreducible* $\mathrm{U}(p)\times\mathrm{U}(r)$*-submodule of* $\bigwedge^j \mathfrak{p}$ *with highest weight* $(r\mathbf{1}_{n_1}, \mathbf{0}_{p-(n_1+n_2)}, -r\mathbf{1}_{n_2}) \otimes (n_2-n_1)\mathbf{1}_r$*. Then*

$$\Gamma_W(\theta^{p,r}(\mu)) = \tau^{\xi'(\mu)}_{\mathrm{U}(p+r)},$$

where $\xi'(\mu) = (\mathbf{a} - l\mathbf{1}_{n_1}, \mathbf{0}_{p+r-(n_1+n_2)}, -(\mathbf{b}^t - l\mathbf{1}_{n_2})) + \frac{n_1-n_2}{2}$ *is the highest weight*[10]*. In fact, such module fist occur in* $\bigwedge^{(n_1+n_2)r}\mathfrak{p}$*.*

3.A A surjectivity result of Helgason

In this section, we prove Lemma 45. The proof is essentially just repeat Shimura's argument[Shi90].

Proof of Lemma 45. We first reduce the problem to the case that H is compact.

Note that, there always exist another real form $\mathfrak{g}_0'$ of $\mathfrak{g}$ such that $\mathfrak{k}_0 \triangleq \mathfrak{h}\cap\mathfrak{g}_0'$ is compact. This can be down as following. Extend σ to an involution on $\mathfrak{g}$. There exist a Cartan involution θ of $\mathfrak{g}$ (view as real Lie algebra) commute with σ. Let $\mathfrak{g}_0' = \mathfrak{g}^{\sigma\theta}$. Now σ is a Cartan involution on $\mathfrak{g}_0'$ and $\mathfrak{k}_0' = {\mathfrak{g}_0'}^{\sigma} = \mathfrak{h}\cap\mathfrak{g}^{\theta}$ is compact with complexification $\mathfrak{h}$.

Note that G is a covering, say $p\colon G \to G_{\mathbb{R}}$, of in an open subgroup $G_{\mathbb{R}}$ of the real points of certain affine algebraic group $G_{\mathbb{C}}$ defined over $\mathbb{R}$ (c.f. [Wal88, Section 2.1]).

Let K_H be the maximal compact subgroup of H respect to Cartan involution θ. Clearly $\mathrm{Lie}(K_H) = \mathfrak{g}_0^{\theta} \subset \mathfrak{k}_0$. Now let G' be the subgroup of $G_{\mathbb{C}}$ generated by $p(K_H)$ and $\exp(\mathfrak{g}_0)$. Then the group generated by

[10] $\xi'(\mu)$ is conjugate to $\xi(\mu)$ under Weyl group action

$p(K_H)$ and $\exp(\mathfrak{k}_0)$ is the maximal compact subgroup K' of G' respect to the Cartan involution σ.

The character ρ of H induce a character of $\mathcal{U}(\mathfrak{h})$. By possibly going to certain covering of K' (by abuse of notation we still call it K' and G' denote the corresponding covering), we may extend ρ to a character of K'.

Therefore, we constructed a real from G' of $\mathfrak{g}$ such that

$$\mathcal{U}(\mathfrak{g})^G = \mathcal{U}(\mathfrak{g})^{G'} \quad \text{and} \quad \mathcal{U}(\mathfrak{g})^H = \mathcal{U}(\mathfrak{g})^{K'}$$

Hence we only have to prove

$$\mathcal{U}(\mathfrak{g})^{G'} \to \mathcal{U}(\mathfrak{g})^{K'}/(\mathcal{J}_\rho \mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})^{K'})$$

is surjective. This is Lemma 71 which we will prove later. □

From now on, let G be a real reductive group. Let $\mathfrak{g}_0$ be its Lie algebra, Let σ be a Cartan involution on $\mathfrak{g}_0$ such that $\mathfrak{g}_0$ decompose into $\mathfrak{k}_0 \oplus \mathfrak{p}_0$ under σ. Let ρ be a character of K. Although Theorem 70 below is proved for connected semi-simple group in [Shi90, Section 2], Shimura's argument also work for our case.

Now we repeat Shimura's proof here. Let $C^\infty(G)$ be the set of C^∞ functions on G. Define

$$C^\infty(\rho) = \{\, f \in C^\infty(G) \mid f(kg) = \rho(k)f(g) \,\}.$$

View $C^\infty(G)$ as a right $\mathcal{U}(\mathfrak{g})$-module where element in $\mathcal{U}(\mathfrak{g})$ act on $C^\infty(G)$ as right invariant differential operator. Let $D(\rho)$ be the subalgebra of $\mathcal{U}(\mathfrak{g})$ which maps $C^\infty(\rho)$ into itself. Let $\mathscr{D}(\rho)$ be the image of $D(\rho)$ in the ring of right invariant differential operators. Then

$$\begin{aligned}
[f \cdot (\mathrm{Ad}_k B)](g) &= \left.\frac{d}{dt}\right|_{t=0} f(\exp(t\mathrm{Ad}_k B)g) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(k \exp(tB)k^{-1}g) \\
&= \rho(k)[f \cdot B](k^{-1}g), \quad \forall f \in C^\infty(\rho), k \in K, B \in \mathfrak{g}_0.
\end{aligned} \tag{3.24}$$

Above equation still true for $B \in \mathcal{U}(\mathfrak{g})$. For any $B \in \mathcal{U}(\mathfrak{g})^K$, $[f \cdot B](g) = \rho(k)[f \cdot B](k^{-1}g)$. Therefore, $\mathcal{U}(\mathfrak{g})^K$ preserve the space $C^\infty(\rho)$, i.e. $\mathcal{U}(\mathfrak{g})^K \subset D(\rho)$.

For any $X \in \mathfrak{k}_0$ and $f \in C^\infty(\rho)$, we have $[f \cdot X](g) = \rho(X)f$. In particular, $fB = \rho(B)f$ for all $B \in \mathcal{U}(\mathfrak{k})$ and $f \in C^\infty(\rho)$.

Let $\mathcal{J}_\rho = \mathrm{Ann}_{\mathcal{U}(\mathfrak{k})}(\rho)$ be the annihilator ideal of ρ in $\mathcal{U}(\mathfrak{k})$.

Let $\psi\colon \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ be the symmetrization map (c.f. [Shi90, Section 1]) characterized by the property that $\psi(X^r) = X^r$ for every $X \in \mathfrak{g}$. ψ is G-equivariant, i.e. $\psi(\mathrm{Ad}(g)B) = \mathrm{Ad}(g)\psi(B)$ for all $B \in \mathcal{S}(\mathfrak{g})$.

Proposition 69 (Proposition 2.1 and Proposition 2.3 [Shi90]). *(1) An element $B \in \mathcal{U}(\mathfrak{g})$ annihilates $C^\infty(\rho)$ if and only if $B \in \mathcal{J}_\rho\mathcal{U}(\mathfrak{g})$.*

(2) $D(\rho) = \mathcal{U}(\mathfrak{g})^K + \mathcal{J}_\rho\mathcal{U}(\mathfrak{g})$.

(3) The natural map of $D(\rho)$ onto $\mathscr{D}(\rho)$ gives an isomorphism of $\mathcal{U}(\mathfrak{g})^K/(\mathcal{U}(\mathfrak{g})^K\cap \mathcal{J}_\rho\mathcal{U}(\mathfrak{g}))$ onto $\mathscr{D}(\rho)$.

(4) The symmetrization map ψ gives a $\mathbb{C}$-linear bijection from $\mathcal{S}(\mathfrak{p})^K$ to $\mathscr{D}(\rho)$.

Proof. It is clear that $\mathcal{J}_\rho\mathcal{U}(\mathfrak{g})$ annihilate $C^\infty(\rho)$. To prove the converse, recall that the map $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k\exp(X)$ is a diffeomorphism. Therefore the map given by

$$h \mapsto (f_h\colon k\exp(X) \mapsto \rho(k)h(X)) \quad \forall h \in C^\infty(\mathfrak{p}_0)$$

is a $\mathbb{C}$-linear bijection between $C^\infty(\mathfrak{p}_0)$ and $C^\infty(\rho)$. Identify $\mathcal{S}(\mathfrak{p})$ with the space of complex coefficient differential operators on $C^\infty(\mathfrak{p}_0)$. Now it is easy to see that

$$[f_h \cdot \psi(B)](e) = [h \cdot B](0) \quad \forall B \in \mathcal{S}(\mathfrak{p}), \tag{3.25}$$

where e is the identity of G. By PBW-theorem,

$$\mathcal{U}_r(\mathfrak{g}) = \bigoplus_{s+t\le r} \psi(\mathcal{S}^s(\mathfrak{k}))\psi(\mathcal{S}^t(\mathfrak{p})). \tag{3.26}$$

Suppose $T \in D(\rho)\cap\mathcal{U}_r(\mathfrak{g})$, $T = \sum_i C_i\psi(B_i)$ with $C_i \in \mathcal{U}(\mathfrak{k})$ and $\{ B_i \}$ a basis of $\mathcal{S}_r(\mathfrak{p})$. If T annihilate $C^\infty(\rho)$, $0 = [f_h \cdot T](1) = \sum_i \rho(C_i)[h \cdot B_i](0)$. Since B_i is linearly independent, by choosing suitable h, we have $\rho(C_i) = 0$ for every i. Therefore $T \in \mathcal{J}_\rho\mathcal{U}(\mathfrak{g})$. This proves (1).

By equation 3.24, $[f \cdot (\mathrm{Ad}_k B)](g) = [f \cdot B](g)$ if $B \in D(\rho)$ and $f \in C^\infty(\rho)$, i.e. $B - \mathrm{Ad}_k B$ annihilate $C^\infty(\rho)$. Take the Haar measure on K such that the total measure of K is 1. Then $B - \int_K \mathrm{Ad}_k B\, dk \in J_\rho\mathcal{U}(\mathfrak{g})$. This proves (2). (3) is clear from (1) and (2).

Again by PBW-theorem, $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{k}) \otimes \psi(\mathcal{S}(\mathfrak{p}))$. For any $X \in \mathcal{U}(\mathfrak{g})$, there is $C_i \in \mathcal{U}(\mathfrak{k})$ and $B_i \in \mathcal{S}(\mathfrak{p})$ such that $X = \sum_i C_i\psi(B_i)$. Now $f \cdot X =$

$\sum_i \rho(C_i)f \cdot B_i$ and $f \cdot (\mathrm{Ad}_k X) = \sum_i \rho(C_i) f \cdot (\mathrm{Ad}_k B_i)$. Hence, by averaging on K, one can see $\mathcal{S}(\mathfrak{p})^K$ is surjective to $\mathscr{D}(\rho)$ since $\mathcal{U}(\mathfrak{g})^K$ is surjective to $\mathscr{D}(\rho)$. By equation (3.25) and suitable choice of h, one conclude that ψ is injective. This proves (4). □

Theorem 70 (Theorem 2.4 [Shi90]). *Suppose that*

$$\psi(\mathcal{S}_r(\mathfrak{g})^G) + \mathfrak{k}\mathcal{U}(\mathfrak{g}) \supset \psi(\mathcal{S}_r(\mathfrak{p})^K) \quad \forall r \in \mathbb{N}. \tag{3.27}$$

Then $\mathcal{Z}(\mathfrak{g})$ *maps onto* $\mathcal{U}(\mathfrak{g})^K/(\mathcal{J}_\rho \mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})^K)$.

Proof. Since $\mathcal{U}(\mathfrak{g})^K/(\mathcal{J}_\rho\mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})^K) \cong \mathscr{D}(\rho)$, we will prove $\mathcal{Z}(\mathfrak{g})$ maps onto $\mathscr{D}(\rho)$. Prove by induction on r that $\psi(\mathcal{S}_r(\mathfrak{p})^K)$ action on $C^\infty(\rho)$ can be obtained from $\psi(\mathcal{S}_r(\mathfrak{g})^G)$. Let $B \in \psi(\mathcal{S}_r(\mathfrak{p})^K)$. By (3.27), there is $T \in \psi(\mathcal{S}_r(\mathfrak{g})^G)$ such that $B - T \in \mathfrak{k}\mathcal{U}(\mathfrak{g})$. By PBW-theorem, $\mathcal{U}_r(\mathfrak{g}) = \psi(\mathcal{S}_r(\mathfrak{p})) \oplus \mathfrak{k}\mathcal{U}_{r-1}(\mathfrak{g})$. Therefore $B - T \in \mathfrak{k}\mathcal{U}_{r-1}(\mathfrak{g})$. Applying (3.26) to $\mathcal{U}_{r-1}(\mathfrak{g})$, we have $B - T = \sum_i Q_i \psi(P_i)$ with $P_i \in \mathcal{S}_{r-1}(\mathfrak{p})$ and $Q_i \in \mathcal{U}(\mathfrak{k})$. It is clear that $B - T$ act on $C^\infty(\rho)$ by $\sum_i \rho(Q_i)\psi(P_i)$. Note that P_i and $\int_K \mathrm{Ad}_k P_i \, dk \in \mathcal{S}_{r-1}(\mathfrak{p})^K$ has same action on $C^\infty(\rho)$ which can be obtained from $\psi(\mathcal{S}_{r-1}(\mathfrak{g})^G)$ by the induction hypothesis. Therefore B action can be obtained from $\psi(\mathcal{S}_r(\mathfrak{g})^G)$ and this complete the proof. □

Helgason (Proposition 7.4 and Theorem 7.5 (i) in [Hel64]) essentially proved equation (3.27) is true if G is connected classical semisimple Lie group.A later paper of Helgason [Hel92] implies equation (3.27) is true for connected semisimple Lie group if and only if G does not contain any simple factor of the following exceptional types: $\mathfrak{e}_{6(-14)}, \mathfrak{e}_{6(-26)}, \mathfrak{e}_{7(-25)}$ and $\mathfrak{e}_{8(-24)}$.

Now extend this result to the reductive dual pair setting as following. Let $\mathfrak{g}_0$ be a reductive Lie algebra with all simple factors classical. Let G^0 be the connected component of G and K^0 the connected component of K, which is also a maximal compact subgroup of G^0. We have $\mathfrak{g}_0 = \mathfrak{c}_0 \oplus \mathfrak{g}_0^s$ where $\mathfrak{g}_0^s = [\mathfrak{g}_0, \mathfrak{g}_0]$ is the semisimple part of $\mathfrak{g}_0$ and $\mathfrak{c}_0$ is the center of $\mathfrak{g}_0$. $\mathfrak{g}_0^s$ has Cartan decomposition $\mathfrak{g}_0^s = \mathfrak{k}_0^s \oplus \mathfrak{p}_0^s$. Moreover $\mathfrak{c}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{t}_0 = \mathfrak{c}_0 \cap \mathfrak{k}_0$ and $\mathfrak{a}_0 = \mathfrak{c}_0 \cap \mathfrak{p}_0$.

Now $\mathcal{S}(\mathfrak{g}) = \mathcal{S}(\mathfrak{c}) \otimes \mathcal{S}(\mathfrak{g}^s)$ and $\mathcal{S}(\mathfrak{p}) = \mathcal{S}(\mathfrak{a}) \otimes \mathcal{S}(\mathfrak{p}^s)$. Since G^0 is generated by $\exp(\mathfrak{g}_0)$,

$$\begin{aligned} \mathcal{S}(\mathfrak{g})^{G^0} =& \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0} = \mathcal{S}(\mathfrak{c}) \otimes \mathcal{S}(\mathfrak{g}^s)^{\mathfrak{g}^s}; \\ \mathcal{S}(\mathfrak{p})^{K^0} =& \mathcal{S}(\mathfrak{p})^{\mathfrak{k}_0} = \mathcal{S}(\mathfrak{a}) \otimes \mathcal{S}(\mathfrak{p}^s)^{\mathfrak{k}_0^s}. \end{aligned}$$

Therefore, by Helgason's result,

$$\psi(\mathcal{S}_r(\mathfrak{g})^{\mathfrak{g}_0}) + \mathfrak{k}\mathcal{U}(\mathfrak{g}) \supset \psi(\mathcal{S}_r(\mathfrak{p})^{\mathfrak{k}_0}).$$

Since G is generated by K and $\exp(\mathfrak{g}_0)$, $\mathcal{S}_r(\mathfrak{g})^G = (\mathcal{S}_r(\mathfrak{g})^{\mathfrak{g}_0})^K$. Averaging over K we conclude equation (3.27) holds. In fact, for any $P \in \mathcal{S}_r(\mathfrak{p})^K \subset \mathcal{S}_r(\mathfrak{p})^{\mathfrak{k}_0}$, there exist $P' \in \mathcal{S}_r(\mathfrak{g})^{\mathfrak{g}_0}$ such that $\psi(P) = \psi(P') + XB$ with $X \in \mathfrak{k}$ and $B \in \mathcal{U}(\mathfrak{g})$. We have

$$\psi(P) = \psi(\int_K \mathrm{Ad}_k P\, dk) = \int_K [\mathrm{Ad}_k \psi(P') + (\mathrm{Ad}_k X)(\mathrm{Ad}_k B)\,] dk \subset \psi(\mathcal{S}_r(\mathfrak{g})^G) + \mathfrak{k}\mathcal{U}(\mathfrak{g})$$

since $\int_K \mathrm{Ad}_k P'\, dk \in \mathcal{S}_r(\mathfrak{g})^G$.

Combine above argument with Theorem 70, we get following lemma and finished the prove of Lemma 45.

Lemma 71. *Lemma 45 holds when H is compact.*

Chapter 4

Lifting of invariants under local theta correspondence

4.1 Introduction

Irreducible representations of real reductive groups are parametrized by Langlands parameters, which could be think as the finest invariants containing complete informations. But, Langlands parameters are difficult to calculate, especially for theta lifts. On the other hand, we can define coarser invariants. These invariants not only are easier to be calculated in some cases but also directly provide some informations about a representation. Here, we study some invariants reflect the “size” and K-spectrums of representations.

For compact group, the dimension of an irreducible module is an invariant which measures its “size”. For an infinite-dimensional irreducible module of a non-compact real reductive group, by taking a filtration of the module, one can define invariants by commutative ring theory, such as the Gelfand-Kirillov dimension, the associated variety and the associated cycle, estimating the growth of dimensions when degree increases.

To state the main theorems we briefly review the definition of associated cycle. Let V be a finitely generated admissible $(\mathfrak{g}, K)$-module. Taking a “good filtration” on V, which is K-invariant, the graded module $\mathrm{Gr}\, V$ will be a finitely generated $\mathcal{S}(\mathfrak{p})$-module with compatible K-action. The associated cycle $\mathrm{AC}(V)$ of V is defined to be the $\mathbb{N}$-linear combination of the isolated primes of $\mathrm{Gr}\, V$, where the coefficient of each prime is its multiplicity in $\mathrm{Gr}\, V$. Since $\mathrm{Gr}\, V$ has K-equivariant action, $\mathrm{AC}(V)$ is a $\mathbb{N}$-linear combinations of $K_{\mathbb{C}}$-coadjoint orbits in $\mathfrak{p}^*$ (c.f. [Vog91] or Section 2.5). We can view the definition of $\mathrm{AC}(V)$ geometrically. The localization of $\mathrm{Gr}\, V$

is a $K_{\mathbb{C}}$-equivariant coherent sheaf, say $\mathscr{V}$ on $\mathfrak{p}^*$. The $K_{\mathbb{C}}$-coadjoint orbits appeared in AC(V) are the open $K_{\mathbb{C}}$-orbits in the support of $\mathscr{V}$ and their multiplicities are the dimension of the fiber at points in these orbits (may have to taking some graded module of $\mathscr{V}$). Moreover, the fiber at a point in general position is a representation of its isotropy subgroup, which is called the isotropy representation at this point. Clearly, isotropy representations contains even finer information than associated cycles.

The associated cycles of theta lifts have been calculated by several authors. The result related to our later discussion includes[1]: theta lifts of unitary characters [Tra04] [Yan11], unitary lowest weight modules constructed from compact dual pairs [NOT01][Yam01] and their theta lifts in the stable range [NZ04]. These results all suggests that theta lifting and taking associated cycle "commute" with each other. For theta lifting of a stable range dual pair (G, G') with G' the smaller member, the notion of theta lifting (c.f. Section 2.3.7.3) of unipotent coadjoint orbits is well defined[2] and gives a map

$$\theta\colon \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*) \to \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*),$$

from the sets of unipotent orbits (or their formal sums) in $\mathfrak{p}'^*$ to which in $\mathfrak{p}^*$. Recall that ρ'^* denote contragredient of ρ'. Now "commute" means, for $\widetilde{G}'$-module ρ',

$$\mathrm{AC}(\theta(\rho')) = \theta(\mathrm{AC}(\rho'^*)). \tag{4.1}$$

Nishiyama and Zhu [NZ04] verified above equation for stable range theta lifts of unitary lowest weight modules. Our calculation will show that, for unitary lowest weight modules, (4.1) also could be true outside stable range sometimes (c.f. the Case I in Section 4.6). On the other hand, we also have examples outside stable range violate (4.1). In fact, for Case II in Section 4.6, (4.44) implies that multiplicity of $\theta^{p,r}(L_{\widetilde{G}'}(\mu))$ is equal to $\dim(A_s \otimes \tau)^{G^{r'-r}}$ which is usually not equal to the multiplicity of $L_{\widetilde{G}'}(\mu)$ described in Section 4.4.

Now we summarise our theorems in the following form. For the precise statements of theorems in each cases, see Theorem 80, Theorem 81 and Theorem 84.

Theorem. *Let (G, G') be a real reductive dual pair in certain good range*[3]

[1] May not be complete.

[2] Out side the stable range, it may not be well defined for certain nilpotent orbit.

[3] Stable range for finte dimensional unitary module; Case I in Table 4.2 for unitary lowest weight module.

and $\rho' \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$ *be a finite dimensional unitary module or an unitary lowest weight module. Then,*

(i) there is an nilpotent $K'_{\mathbb{C}}$*-orbit* $\mathcal{O}' \subset \mathfrak{p}'^*$ *such that the associated variety* $\mathcal{V}(\rho'^*) = \overline{\mathcal{O}'}$*;*

(ii) $\mathcal{O}'$ *has theta lift* $\mathcal{O} = \theta(\mathcal{O}') \subset \mathfrak{p}^*$ *and* $\mathcal{V}(\theta(\rho')) = \overline{\mathcal{O}}$*;*

(iii) for $x' \in \mathcal{O}$*, let* $K'_{x'}$*-module* $\chi_{x'}$ *be the isotropy representation of* $\operatorname{Gr}\rho'^*$*; there is a point* $x \in \mathcal{O}$ *and a homomorphism* $\alpha\colon K_x \to K'_{x'}$*, such that the isotropy representation of* $\operatorname{Gr}\theta(\rho')$ *at point* x *is*

$$\chi_x \cong \chi_{x'} \circ \alpha,$$

up to a twisting;

(iv) in particular, the multiplicity of associated cycle is preserved, i.e.

$$\mathrm{AC}(\theta(\rho')) = \theta(\mathrm{AC}(\rho'^*)).$$

Remark:

1. As suggested by the formulation of the theorem, instead of calculating the associated cycle directly, we actually calculated the isotropy representation of the theta lift $\theta(\rho')$ in terms of the isotropy representation of ρ'.

2. For finite dimensional unitary representations, $\mathcal{O}'$ is the zero orbit and $\mathrm{AC}(\rho') = \mathrm{AC}(\rho'^*) = \dim\rho' \cdot \{\,0\,\}$.

3. When ρ' is an unitary character, part (ii) and (iv) is more or less known to experts, see [Prz91][Prz93] and part (iii) seems implicitly assumed in [Yan11]. In Section 4.5, we will give an unified proof for all dual pairs in stable range.

4. For an unitary lowest weight module, which is a theta lift of a finite dimensional unitary module, its associated cycle was calculated in [NOT01] and its isotropy representation was calculated in [Yam01]. The corresponding orbit is called holomorphic orbits.

5. The theta lifts of holomorphic orbits were studied in [NOZ06]. The associated cycle of stable range theta lifts of unitary lowest weight modules were calculated in [NZ04]. In Section 4.6, we will calculate their isotropy representations (possibly outside the stable range) and recover Nishiyama-Zhu's results.

6. Our calculation is based on a geometric method inspired by Nishiyama and Zhu's work [NZ04].

7. See Remark 1 after Theorem 81 for the twisting of characters.

8. When ρ' is the unitary lowest weight module, ρ' is a unitary highest weight module. It is easy to calculate the invariants of ρ'^* in terms of invariants of ρ', see Remark 5 after Theorem 84.

Now we highlight the strategies to obtain theorems in above form. We first identify the graded module of theta lift with the space of E-invariant sections of an E-equivariant sheaf, see (4.11), (4.28) and (4.34), where E is a reductive group. The nilpotent orbit will correspond to an E-quotient of the base space. Then we calculate the isotropy representation in terms of the E-invariants of the fiber of the quotient map, see (4.13), (4.30) and (4.41). Under a "good range", its fiber will be a single orbit under the isotropy group action. Then we obtain (iii) by some form of reciprocity formula, see Lemma 79. For some the bad cases, i.e. non-Tube type domain for unitary lowest weight module and Case II for its theta lift, we have to adapt one more step of fibration.

Furthermore, Vogan [Vog91, Theorem 4.11] show that the K-spectrum of an irreducible $(\mathfrak{g}, K)$-modules is controlled by the space of global sections of certain equivariant (algebraic) vector bundle (determined by the isotropy representation) over the open orbit in its associated variety, up to an error come from the boundary. Vogan expect that [Vog91, Conjecture 12.1], for certain pair[4] of orbit and isotropy representation $(\mathcal{O}, \chi)$, there always exits an unipotent representation, say V, such that its K-spectrum is exactly given by the space of global sections, i.e., as K-module,

$$V \cong \mathrm{Ind}_{K_x}^{K_\mathbb{C}} \chi, \tag{4.2}$$

where x is an element in $\mathcal{O}$, K_x is the isotropy group of x, χ is a rational representation of (possible double covering of) K_x.

We will show that some families of representation we mentioned in the Theorem satisfies equation (4.2). Especially, we highlight following three families: (a) stable range theta lifts of unitary characters; (b) Some singular unitary lowest weight modules; and (c) stable range theta lifts of these singular unitary lowest weight moudles.

We obtain these results by geometric method, without using any explicit branching formula. It is worthy of note that, while our approach appears in an algebraic geometric form, the results are essentially based on K-spectrum formulas obtained by analytic constructions back to Howe [How83], Li [Li89] and Huang-Zhu [ZH97] (c.f. Section 2.3.6).

We should point out that: family (a) is calculated in Yang's thesis[Yan11].

[4] Called admissible data. See Section 2.5.

He showed that they are "unipotent" representations attached to nilpotent orbits of height 2. Height means number of columns in the Young diagram[5] parameterization of the nilpotent orbit. We believe that family (c) is new. Some representations in this family which are two-step theta lifts of unitary characters, could be positive examples for Vogan's prediction and should be unipotent representations attached to nilpotent orbits of height 3, in Yang's sense. We omite the verification of the admissibility of orbit data[6], since it is less enlightening[7] before one can proving a theorem in a general setting. See Conjecture 72 and its remarks .

Our results suggest that equations of form (4.2) may not only hold for unipotent representations, but also hold for much more general cases. Theta lifting seems be an efficient tool to construct such representations.

Based on our results of associated cycles, isotropy representations and K-spectrum equations, we would like to pose following conjecture for the general cases.

Conjecture 72. *Suppose (G, G') is under stable range with G' the smaller member and ρ' is a genuine unitary representation of $\widetilde{G}'$. Let $\theta(\rho')$ be its theta lift. Let $\{ \mathcal{O}'_j \}$ be the set of open orbits in $\mathcal{V}(\rho'^*)$ such that*

$$\mathcal{V}(\rho'^*) = \bigcup_j \overline{\mathcal{O}}'_j.$$

Suppose $x'_j \in \mathcal{O}'_j$ and $\chi_{x'_j}$ is the isotropy representation of the isotropy subgroup $K'_{x'_j} = \mathrm{Stab}_{K'_{\mathbb{C}}}(x'_j)$. Then,

(i) for each j, there is a point $x_j \in \mathcal{O}_j := \theta(\mathcal{O}'_j)$ and a homomorphism[8] $\alpha_j \colon K_{x_j} \to K'_{x'_j}$ such that the isotropy representation of K_{x_j},

$$\chi_{x_j} \cong \alpha_j^*(\chi_{x'_j}),$$

up to a twisting of certain character; here α_j^* is the functor maps virtual characters of $K'_{x'_j}$ to virtual characters of K_{x_j} by pre-composition, may consider it as the "lifting map" between spaces of isotropy representations;

(ii) if ρ' satisfies the K-spectrum equation (4.2), $\theta(\rho')$ also satisfies the K-spectrum equation.

[5] Nilpotent coadjoint $K_{\mathbb{C}}$-orbits of classical Lie algebra are usually parametrized by signed Young-diagram. One can write done the explicit parameter for all nilpotent orbits appeared here. The theta lifting of nilpotent orbits is corresponding to a column adding operation on the diagram. But, we omit the descriptions of Young-diagrams here, since they are irrelevant to our geometric approach.

[6] It is down in [LMT11a] for pair $(\mathrm{O}(p,q), \mathrm{Sp}(2n, \mathbb{R}))$.

[7] At least to the author.

[8] See Lemma 17 for its definition.

As a consequence of (i), theta lifting and taking associated cycle commutes:

$$\mathrm{AC}(\theta(\rho')) = \theta(\mathrm{AC}(\rho'^{*})). \tag{4.3}$$

Remark:

1. Nishiyama and Zhu speculated (4.3) (without contragredient) in their paper [NZ04], and it is probably also be expected by other experts.

2. Currently, there is no evidence that above guess is true in the generality that the associated variety of the original representation contains several open orbits.

3. Above conjecture has an inductive natural. Philosophically, theta correspondence should be able to construct unipotent representations of classical groups by an iterating process[9], for example see [Bry03] and [He07]. The author hope that above speculation is at least true for iterated stable range theta lifts of unitary characters. It would provide more evidences to this philosophy and may contribute to a better understanding of unipotent representations.

4. The proof of K-spectrum equation (4.2) may rely on some deep analytic results, especially some automatic continuity theorem in a general setting.

5. See Section 2.3.7.3 for the construction of theta lifts of nilpotent orbits and the map $\alpha_j\colon K_{x_j} \to K'_{x_j}$.

The organization of this chapter is as follows. In Section 4.2, we will study natural filtrations on theta lifts and show they are "good filtrations". After that, we supply some technical lemmas. Then, we investigate the isotropy representations together with K-spectrum equations (4.2) for unitary lowest weight modules, theta lifting of unitary characters and theta lifts of unitary lowest weight modules in Section 4.4, Section 4.5 and Section 4.6 respectively.

4.2 Natural filtrations on theta lifts

Let (G, G') be a non-compact reductive dual pair in $\mathrm{Sp} := \mathrm{Sp}(W)$. A maximal compact subgroup U of $\mathrm{Sp}(W)$ is fixed by choosing a compatible complex structure $W^{\mathbb{C}}$ of W, i.e. the symplectic form is the imaginary part of the positive definite Hermition form on $W^{\mathbb{C}}$. U could be chosen such that $K = G \cap \mathrm{U}$ (resp. $K' = G' \cap \mathrm{U}$) is a maximal compact subgroup of G (resp. G').

[9] Here, we studied one-step and two-step theta liftings.

Let $\mathscr{Y}$ be the Fock model of the oscillator representation of $\widetilde{\mathrm{Sp}}$. Then $\mathscr{Y}$ is isomorphic to the space of polynomials on $W^{\mathbb{C}}$. Suppose $\rho' \in \mathscr{R}(\mathfrak{g}', \widetilde{K}'; \mathscr{Y})$, let ρ be its maximal Howe quotient $\Theta(\rho')$ or its theta lift $\theta(\rho')$. Let

$$\eta \colon \mathscr{Y} \to \rho \otimes \rho' \tag{4.4}$$

be the unique (up to scaler) $(\mathfrak{g}, \widetilde{K}) \times (\mathfrak{g}', \widetilde{K}')$-equivariant quotient map .

Let σ' be the lowest degree $\widetilde{K}'$-type of ρ'. Let ν be the map defined by composition of η and the projection to a fixed one-dimensional subspace of the σ'-isotypic component.[10] Note that $\mathscr{Y}$ has a natural filtration $\{ \mathscr{Y}_j \}$, where $\mathscr{Y}_j$ is the space of polynomials of by degree less than or equal to j. Let $j_0 = \deg \sigma' = \min \{ j \mid \eta(\mathscr{Y}_j) \neq 0 \}$. We will choose the one-dimensional subspace in σ'-isotypic component such that it has lowest degree, i.e. we require[11] $\nu(\mathscr{Y}_{j_0}) \neq 0$. The filtration on $\mathscr{Y}$ induce a filtration on ρ via ν. We call the filtration on ρ the *natural filtration*.

Note that $\mathscr{Y} = \mathscr{Y}_{\mathrm{even}} \oplus \mathscr{Y}_{\mathrm{odd}}$ as irreducible $(\mathfrak{sp}, \widetilde{\mathrm{U}})$-module, where $\mathscr{Y}_{\mathrm{even}}$ and $\mathscr{Y}_{\mathrm{odd}}$ are direct sum of even degree and odd degree polynomials respectively. η vanish[12] on one of $\mathscr{Y}_{\mathrm{even}}$ and $\mathscr{Y}_{\mathrm{odd}}$. In particular, this means

$$\nu(\mathscr{Y}_{2j+j_0}) = \nu(\mathscr{Y}_{2j+j_0+1}). \tag{4.5}$$

Therefore we will concentrate on the degrees with same parity of j_0.

Let

$$F_j = \nu(\mathscr{Y}_{2j+j_0}) \qquad G_j = \mathcal{U}_j(\mathfrak{g})\nu(\mathscr{Y}_{j_0})$$

A priori, F_j and G_j are two filtrations on ρ. We review the following lemma, claim that they are same filtration.

Lemma 73 (c.f. Section 3.3 [NZ04], Lemma 4.1 [LMT11a]). *We have*

$$F_j = G_j.$$

[10]This procedure is achievable: picking up the σ'-isotypic component by integrating against its character over $\widetilde{K}$; then project to a one-dimensional subspace by a combination of action of $\mathcal{U}(\mathfrak{g}')^{K'} \times \widetilde{K}'$ action, since $\mathcal{U}(\mathfrak{g}')^{K'} \times K'$ act irreducibly on the finite dimensional σ'-isotypic component.

[11]Actually lowest degree K-type is multiplicity one [He00, Theorem 13.5], so whole σ'-isotypic component is in the image of joint harmonics under η. But, in fact, the later statement and argument on filtrations still valid for any $\widetilde{K}'$ by providing this requirement.

[12]Without going though Howe's proof of duality theorem, one can prove the claim as following: by the Howe-duality, $\Theta(\rho')$ has a irreducible quotient map; if η is non-vanishing on both components, pre-composite with projection maps will lead to two different non-zero quotient; this lead a contradiction.

Therefore, F_j is a good filtration of ρ, since G_j is good by definition.

Before proving the lemma, recall the diamond dual pairs introduced by Howe [How89b] (c.f. Figure 2.1). We follow Howe's notation [How89b]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ under Cartan decomposition. Let M be the subgroup of Sp such that (M, K') form a compact dual pair. Now M is Hermitian symmetric, $\mathfrak{m} = \mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{2,0} \oplus \mathfrak{m}^{(0,2)}$. Here $\mathfrak{m}^{(1,1)}$ correspond to the maximal compact subgroup of $\mathfrak{m}$ and act on $\mathscr{Y}$ by degree preserving operator. $\mathfrak{m}^{(2,0)}$ and $\mathfrak{m}^{(0,2)}$ are abelian Lie subalgebra of $\mathfrak{m}$ acting on $\mathscr{Y}$ by K'-invariant quadratic polynomials and differential operators respectively.

Fact 3 in Howe's paper[How89b] states that in $\mathfrak{sp}$, we have

$$\mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)} = \mathfrak{p} \oplus \mathfrak{m}^{(0,2)}. \tag{4.6}$$

The projection of $\mathfrak{p}$ to $\mathfrak{m}^{(2,0)}$ with respect to the left hand side of (4.6) is a K-isomorphism. We will identify $\mathfrak{p}$ with $\mathfrak{m}^{(2,0)}$ via this projection.

Proof of Lemma 73. The map ν factors through the σ'-type $\widetilde{K}'$-covariant space $\mathscr{Y}_{\sigma'}$ of $\mathscr{Y}$. Let $\mathscr{Y}(\sigma')$, $\mathscr{Y}^d(\sigma')$ and $\mathscr{Y}_j(\sigma') = \oplus_{d \le j} \mathscr{Y}^d(\sigma)$ be the σ'-isotypic components of $\mathscr{Y}$, $\mathscr{Y}^d$ and $\mathscr{Y}_j$ respectively. Since $\widetilde{K}'$ action is reductive on $\mathscr{Y}$ and preserves degree, $\mathscr{Y}(\sigma')$ maps bijectively onto the covariant $\mathscr{Y}_{\sigma'}$. Moreover $\mathscr{Y}^d(\sigma') = \mathscr{Y}^d \cap \mathscr{Y}(\sigma')$ and $\mathscr{Y}_j(\sigma') = \mathscr{Y}_j \cap \mathscr{Y}(\sigma')$. Hence

$$\nu(\mathscr{Y}_{2j+j_0}(\sigma')) = F_j. \tag{4.7}$$

Let $\mathcal{H}(\sigma')$ be the σ'-isotypic component in the space of harmonic $\mathcal{H}(K')$ for $\widetilde{K}'$. By [How89a], we have $\mathcal{H}(\sigma') \subset \mathscr{Y}^{j_0}$ and

$$\mathscr{Y}(\sigma') = \mathcal{U}(\mathfrak{m}^{(2,0)})\, \mathcal{H}(\sigma'). \tag{4.8}$$

Since $\mathfrak{m}^{(2,0)}$ act by degree two polynomials, $\mathscr{Y}^j(\sigma') = 0$ if $j \not\equiv j_0 \pmod 2$. This leads yet another proof of (4.5). It follows from (4.7) and (4.8) that $\nu(U_j(\mathfrak{m}^{(2,0)})\, \mathcal{H}(\sigma')) = F_j$.

Prove $G_j = F_j$ by induction. First, we have $G_0 = F_0$ by definition; Now suppose $G_{j-1} = F_{j-1} = \nu(\mathscr{Y}')$, where $\mathscr{Y}' = \mathscr{Y}_{2(j-1)+j_0}$. Since $G_j \subseteq F_j$, it is suffices to show that $F_j \subseteq G_j$. By (4.6), $\mathscr{Y}' + \mathfrak{m}^{(2,0)}\mathscr{Y}' = \mathscr{Y}' + \mathfrak{p}\mathscr{Y}'$. Hence

$$\begin{aligned} F_j &= \nu(U_j(\mathfrak{m}^{(2,0)})\mathcal{H}(\sigma')) \subseteq \nu(\mathscr{Y}' + \mathfrak{m}^{(2,0)}\mathscr{Y}') = \nu(\mathscr{Y}' + \mathfrak{p}\mathscr{Y}') = \nu(\mathscr{Y}') + \mathfrak{p}\nu(\mathscr{Y}') \\ &= G_{j-1} + \mathfrak{p}G_{j-1} \subseteq G_j. \end{aligned}$$

This completes the proof of the lemma. □

In the rest of the section, we adapt the notation in Section 4.6 and study filtrations on $\Theta^{p,r}(\mu)$. Now ρ' is a character. Fix a linear functional $\lambda \in \tau^{\mu}_{\widetilde{K}^{r'}}$ and let pr be the projection to $\widetilde{K}^{r'}$-invariant space. Define

$$\iota\colon \Theta^{p,q}(\rho') \to \Theta^{p,q}(\rho') \otimes \tau^{\mu}_{\widetilde{K}^{r'}} \to \left(\Theta^{p,q}(\rho') \otimes \tau^{\mu}_{\widetilde{K}^{r'}}\right)^{\widetilde{K}^{r'}} \cong \Theta^{p,r}(\mu)$$

by $v \mapsto \mathrm{pr}(v \otimes \lambda)$. Let $\nu' = \iota \circ \nu$.

Let l_0 be the minimal l such that $\iota(F_l) \neq 0$. Let j_0 be the minimal j such that $\nu'(\mathscr{Y}) \neq 0$. Define

$$F'_j = \iota(F_{j+l_0}) = \nu'(\mathscr{Y}_{2j+j_0}) \quad \text{and} \quad G'_j = \mathcal{U}_j(\mathfrak{g}_1)\iota(F_{l_0}) = \mathcal{U}_j(\mathfrak{g}_1)\nu'(\mathscr{Y}_{j_0}).$$

Here F'_j is the filtration of $\Theta^{p,r}(\mu)$ inherited form the natural filtration of $\Theta^{p,q}(\rho')$. The following lemma shows that F'_j and G'_j are same filtration.

Lemma 74 (Lemma 7.1,[LMT11a])**.** *We have $F'_j = G'_j$. Therefore, F'_j is a good filtration of $\Theta^{p,r}(\mu)$.*

Proof. Clearly $G'_j \subseteq F'_j$. It remains to show that $G'_j \supseteq F'_j$.

We decompose $W = W' \times W''$ where W' and W'' are symplectic spaces associated with pair $(G^{p,r}, G')$ and $(K^{r'}, G')$ respectively. The Fock model has decomposition $\mathscr{Y} = \mathscr{Y}' \otimes \mathscr{Y}''$ with $\mathscr{Y}' = \mathbb{C}[W'^{\mathbb{C}}]$ and $\mathscr{Y}' = \mathbb{C}[W''^{\mathbb{C}}]$. The Lie algebra $\mathfrak{g}'$ acts on both $\mathscr{Y}' \otimes \mathscr{Y}''$ diagonally. Denote $\mathscr{Y}'_d$ (resp. $\mathscr{Y}''_d$) the natural filtration on $\mathscr{Y}'$ (resp. $\mathscr{Y}''$). Clearly,

$$\mathscr{Y}_d = \sum_{a+b=d} \mathscr{Y}'_a \otimes \mathscr{Y}''_b. \tag{4.9}$$

Decompose $\mathscr{Y}'' = \bigoplus_{\mu'} L^*_{\widetilde{G}'}(\mu') \otimes (\tau^{\mu'}_{\widetilde{K}^{r'}})^*$ under $(K^{r'}, G')$ action. Let $\mathrm{pr}_2\colon \mathscr{Y}'' \to L^*_{\widetilde{G}'}(\mu) \otimes (\tau^{\mu}_{\widetilde{K}^{r'}})^*$ be the projection to $(\tau^{\mu}_{\widetilde{K}^{r'}})^*$ isotypic component, $\nu_1\colon \mathscr{Y}' \to \Theta^{p,r}(\mu) \otimes (L_{\widetilde{G}'}(\mu) \otimes \rho')$ be the unique non-zero quotient map by Howe and $\Pi\colon (L_{\widetilde{G}'}(\mu) \otimes \rho') \otimes L^*_{\widetilde{G}'}(\mu) \to \rho'$ be the $\mathfrak{g}'$-equivariant paring.

In the proof of Lemma 62, we have shown that the following diagram commutes,

$$\begin{array}{ccccc}
\mathscr{Y} & \xrightarrow{\nu} & \Theta^{p,q}(\rho') & \xrightarrow{\iota} & \Theta^{p,r}(\mu) \\
\downarrow{\cong} & & & & \uparrow \\
\mathscr{Y}' \otimes \mathscr{Y}'' & \xrightarrow[\nu_1 \otimes \mathrm{pr}_2]{} & \Theta^{p,r}(\mu) \otimes (L_{\widetilde{G}'}(\mu) \otimes \rho') \otimes L^*_{\widetilde{G}'}(\mu) \otimes (\tau^{\mu}_{\widetilde{K}^{r'}})^* & \xrightarrow[\mathrm{id}\otimes\Pi\otimes\mathrm{id}]{} & \Theta^{p,r}(\mu) \otimes (\tau^{\mu}_{\widetilde{K}^{r'}})^*
\end{array}$$

i.e. ν' factor through $(\mathrm{id} \otimes \Pi \otimes \mathrm{id}) \circ (\nu_1 \otimes \mathrm{pr}_2)$.

Let σ' be the K'-type of lowest degree for $L_{\widetilde{G}'}(\mu) \otimes \rho'$. Let j'_0 be the degree of σ' in $\mathscr{Y}'$ and let j''_0 be the degree of $\tau^{\mu}_{\widetilde{K}^{r'}}$ in $\mathscr{Y}''$. Then, $j_0 = j'_0 + j''_0$ by (4.9). Since $\mathrm{pr}_2(\mathscr{Y}''_{j''_0}) = \sigma'^* \otimes \tau^{\mu}_{\widetilde{K}^{r'}}$, $\nu'(\mathscr{Y}'_{2j+j'_0} \otimes \mathscr{Y}''_{j''_0}) = G'_j$ by Lemma 73.

Let $\epsilon = 0, 1$. Then $\mathrm{pr}_2(\mathscr{Y}''_{2j''+j''_0+\epsilon}) = \mathrm{pr}_2(\mathcal{U}_{j''}(\mathfrak{p}')\mathscr{Y}''_{j''_0})$, by [How89a]. Now we prove the theorem by induction. The base case is true by definition. Now assume it is hold for $j-1$. Since the maps Π is $\mathfrak{g}'$-equivariant and $\mathfrak{p}'$ act by scaler[13],

$$\begin{aligned}\nu'(\mathscr{Y}'_{a+j'_0} \otimes \mathscr{Y}''_{2b+j''_0+\epsilon}) =& \nu'(\mathscr{Y}'_{a+j'_0} \otimes \mathcal{U}_b(\mathfrak{p}')\mathscr{Y}''_{j_0}) \\ =& \nu'(\mathcal{U}_b(\mathfrak{p}')\mathscr{Y}'_{a+j'_0} \otimes \mathscr{Y}''_{j_0} + \mathscr{Y}_{a+2b+j_0-2}) \\ \subseteq& \nu'(\mathscr{Y}'_{a+2b+j'_0} \otimes \mathscr{Y}''_{j_0} + \mathscr{Y}_{a+2b+j_0-2}) \\ \subseteq& G'_{\lfloor a/2 \rfloor + b} + G'_{\lfloor a/2 \rfloor + b-1} \subseteq G'_{\lfloor a/2 \rfloor + b}.\end{aligned}$$

Hence, $F'_j = \nu'(\mathscr{Y}_{2j+j_0}) \subseteq G'_j$ by (4.9). □

4.3 Some technical lemmas

In this section, we will prove some technical lemmas, which will be used freely in our latter discussion. In this section, let field $k = \mathbb{C}$, although these lemmas usually hold for much more general fields, such as algebraically closed fields of characteristic 0.

Following lemma is about the inverse images of associated sheaves. Section 5.17 [Jan87] gives a similar statement for locally trivial quotient $G \to G/H$.

Lemma 75. *Let L be a subgroup of G and V is a finite dimensional H-module. Let $\phi\colon L/(L \cap H) \to G/H$ be the L-equivariant morphism defined by $L \hookrightarrow G$. Assume $\phi(L)$ is a closed(or open) sub-variety of G/H, then*

$$\phi^* \mathscr{L}_V^{G/H} \cong \mathscr{L}_{V|_{L\cap H}}^{L/(L\cap H)}.$$

Proof. Since $\phi^* \mathscr{L}_V^{G/H}$ is a quasi-coherent $(L/(L \cap H), L)$-module with fiber at $e = H/H$ isomorphic to $V|_{L\cap H}$, the equation holds by the equivalence of categories between rational $L \cap H$-modules and $(L/(L \cap H), L)$-modules (c.f. Theorem 39).

Actually, we can construct the isomorphism directly as following. For an affine cover $\{ U' \}$ of $L/(L \cap H)$, we may assume that $\phi(U') \subset U$, where U is certain affine open subset of G/H such that $\mathscr{L}(U) \cong k[U] \otimes V$, since

[13] Act by zero if ρ' is unitary.

$\mathscr{L}_V^{G/H}$ is locally free[14]. Therefore

$$(\phi^*\mathscr{L}_V)(U') = k[U'] \otimes_{k[U]} (k[U] \otimes V) \cong k[U'] \otimes V \cong \mathscr{L}_{V|_{L\cap H}}^{L/(L\cap H)}(U')$$

gives the isomorphism between sheaves. □

Now let X be an affine variety with G action, G has an open orbit $\mathcal{O}$ in X generated by $x \in X$. Let $H = \mathrm{Stab}_G(x)$ be the stabilizer of $x \in X$. So we have following commutative diagram of inclusions.

$$\begin{array}{ccccc} G/H & = & \mathcal{O} & \xhookrightarrow{i_{G/H}} & X \\ & & \uparrow & \nearrow_{i_x} & \\ & & x & & \end{array}$$

Let A be a $(k[X], G)$-module and $\mathscr{A}$ be its associated qusi-coherent sheaf on X. The fiber of $\mathscr{A}$ at x, $V = i_x^*\mathscr{A} = A/\mathcal{I}_x A$, is a rational H-module[15]. Let $\mathscr{L}$ be the associated qusi-coherent sheaf on G/H. Then by equivalence of categories, $i_{G/H}^*\mathscr{A} \cong \mathscr{L}$. Now we have a natural map $\mathscr{A} \to (i_{G/H})_*(i_{G/H})^*\mathscr{A} = (i_{G/H})_*\mathscr{L}$ and it induce a map between their global sections:

$$\varrho\colon A \to \mathrm{Ind}_H^G V = ((i_{G/H})^*\mathscr{L})(X).$$

Since X is affine, following lemma is trivial from the equivalence of categories between the category of $k[X]$-modules and the category of qusi-coherent sheaves on X[Har77, Corollay 5.5]. We highlight it since we will use it later.

Lemma 76. *If $\varrho\colon A \to \mathrm{Ind}_H^G V$ an isomorphism, $\mathscr{A} \cong (i_{G/H})_*\mathscr{L}$.*

Consider following fiber product, where $f\colon Y \to X$ is a morphism between varieties such that X is affine, $x \in X$ is a closed point and F_x is the scheme theoretical fiber of x.

$$\begin{array}{ccc} F_x & \xrightarrow{i_{F_x}} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_x} & X \end{array}$$

We have following lemma of fibers.

Lemma 77 ([Har77, Chapter III, Corollary 9.4])**.** *Suppose that $\mathscr{F}$ is a quasi-coherent sheaf on X and $k(x)$ is the residual field of x. Then there*

[14]See the proof of [CPS83, Lemma 2.5]

[15]$\mathcal{I}_x$ is the ideal correspond to x

are natural homomorphism

$$\mathscr{F}(Y) \otimes_{k[X]} k(x) = i_x^* f_* \mathscr{F} \cong g_* i_{F_x}^* \mathscr{F} = (i_{F_x}^* \mathscr{F})(F_x).$$

Now, we state a lemma for the reducibility of the fiber. It is familiar to algebraic geometer.

Lemma 78. *Let $\phi\colon Y \to X$ be a G-equivariant morphism between varieties. Suppose $x \in X$ generate an open G-orbit in the image of ϕ. Then its scheme theoretical fiber is reduced. In particular, if X, Y are affine,*

$$k[\phi^{-1}(x)] = k[Y]/\mathcal{I}_x k[Y],$$

where $\mathcal{I}_x$ is the corresponding maximal ideal in $k[X]$.

Proof. Let $\mathcal{O} = G \cdot x$ be the G-orbit. We can reduce the problem to $\overline{\mathcal{O}}$, Y, since reducibility is a local property. Since ϕ is a map between variety Y and $\overline{\mathcal{O}}$ over $\mathbb{C}$, its scheme theoretical fibers are reduced for points in general position. On the other hand, ϕ is $K_{\mathbb{C}}$-equivariant, and $\mathcal{O}$ is open dense in $\overline{\mathcal{O}}$. So the scheme theoretical fiber should be reduced over all points in $\mathcal{O}$, especially for x. □

Now we consider a very special form of reciprocity formula.

Lemma 79. *Let G, H be two affine algebraic group, $\alpha\colon G \to H$ be a homomorphism. Let $G \times_\alpha H$ be the fiber product as in following diagram, i.e. $G \times_\alpha H = \{\, (g, h) \in G \times H \mid \alpha(g) = h \,\}$.*

$$\begin{array}{ccc} G \times_\alpha H & \longrightarrow & H \\ \downarrow & & \Vert \\ G & \xrightarrow{\ \alpha\ } & H \end{array}$$

Let χ be a rational $\alpha(H)$-module. Then as G-module,

$$\left(\mathrm{Ind}_{G \times_\alpha H}^{G \times H} \mathbb{C} \otimes \chi\right)^H \cong \chi \circ \alpha.$$

Proof. Note that, via $g \mapsto (g, \alpha(g))$, G isomorphic to $G \times_\alpha H$, i.e.

$$\mathrm{id} \times \alpha \circ \triangle\colon G \xrightarrow{\ \cong\ } G \times_\alpha H\ .$$

Now the lemma holds by following computation using Theorem 37 (f).

$$
\begin{aligned}
&\left(\mathrm{Ind}_{G\times_\alpha H}^{G\times H}\mathbb{C}\otimes\chi\right)^H \\
=&\left((k[G]\otimes k[H])^{G\times_\alpha H}\otimes\chi\right)^H \\
=&((k[G]\otimes k[H]\times\chi)^H)^{G\times_\alpha H} \\
&\quad\text{(since } G\times_\alpha H \text{ and } H \text{ action commutes)} \\
=&(k[G]\otimes\chi)^{G\times_\alpha H}=(k[G]\otimes\chi)^{\mathrm{id}\times\alpha\circ\triangle G} \\
=&(k[G]\otimes\chi\circ\alpha)^G=\chi\circ\alpha.
\end{aligned}
$$

□

4.4 Isotropy representations of unitary lowest weight modules

In this section, we review the associated cycles and isotropy representations of unitary lowest weight modules obtained by theta lifting. All results in this section was computed in [NOT01] and [Yam01]. Our calculation is essentially same as [Yam01]. We include the material for completeness and for showing that the calculation could be fitted into a general pattern similar to other case.

4.4.1 Statement of the theorem

We retain the notation in Section 2.3.5.2: (G, G') is a compact dual pair, with $G' = K'$ a compact group; G is Hermitian symmetric with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. See Table 4.1

	G	$G' = K'$	Stable range
Case $\mathbb{R}$	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{O}(m)$	$n \geq m$
Case $\mathbb{C}$	$\mathrm{U}(n_1, n_2)$	$\mathrm{U}(m)$	$n_1, n_2 \geq m$
Case $\mathbb{H}$	$\mathrm{O}^*(2n)$	$\mathrm{Sp}(m)$	$n \geq 2m$

Table 4.1: Compact dual pairs for unitary lowest weight modules

Let $L_{\widetilde{G}}(\mu)$ be the theta lift of finite dimensional $\widetilde{G}'$-module $\tau^{\mu}_{\widetilde{G}'}$. So the unitary lowest weight $(\mathfrak{g}, \widetilde{K})$-module

$$
L_{\widetilde{G}}(\mu) \cong \left(\mathscr{Y} \otimes (\tau^{\mu}_{\widetilde{G}'})^*\right)^{\widetilde{K}'} \tag{4.10}
$$

with $(\mathfrak{g}, \widetilde{K})$ act on $\mathscr{Y}$. Recall that $\mathfrak{p}^+$ act on $\mathscr{Y}$ by multiply $K'_{\mathbb{C}}$-invariant quadratic polynomial and $\mathfrak{p}^-$ act on $\mathscr{Y}$ by $K'_{\mathbb{C}}$-invariant quadratic differential operators. Define the filtration on $L_{\widetilde{G}}(\mu)$ by the natural filtration on $\mathscr{Y}$, it is clear that this filtration is good by above description of $\mathfrak{g}$ action. Passing to the graded module, $\mathfrak{p}^-$ action is trivial. Fix a genuine characters $\varsigma \otimes \varsigma'$ of $\widetilde{K} \times \widetilde{K}'$ such that the covering groups act on $\mathscr{Y}$ by linear action tensor with this character.

Recall the moment map

$$\phi\colon W^{\mathbb{C}} \to (\mathfrak{p}^+)^* \quad \text{is defined by} \quad \phi^*\colon \mathcal{S}(\mathfrak{p}^+) \to \mathbb{C}[W^{\mathbb{C}}].$$

and $(\phi, \overline{\phi(W^{\mathbb{C}})})$ is the categorical quotient of $W^{\mathbb{C}}$ under $K'_{\mathbb{C}}$ action (c.f. Section 2.3.7). Twiste the corresponding graded module $\operatorname{Gr} L_{\widetilde{G}}\mu$, define $(\mathcal{S}(\mathfrak{p}^+), K_{\mathbb{C}})$-module

$$\begin{aligned} A :=& \varsigma^* \otimes \operatorname{Gr} L_{\widetilde{G}}(\mu) \cong \left(\mathbb{C}[W^{\mathbb{C}}] \otimes_{\mathbb{C}} \varsigma' \otimes (\tau^{\mu}_{\widetilde{K}'})^*\right)^{\widetilde{K}'} \\ &\cong \left(\mathbb{C}[W^{\mathbb{C}}] \otimes_{\mathbb{C}} \tau^*\right)^{K'_{\mathbb{C}}}, \end{aligned} \tag{4.11}$$

Here

(a)
$$\tau = \varsigma'^* \otimes \tau^{\mu}_{\widetilde{K}'}; \tag{4.12}$$

(b) $\mathfrak{p}^-$ act on $\mathbb{C}[W^{\mathbb{C}}]$ (so on A) trivially;

(c) $f \in \mathfrak{p}^+$ act on $\mathbb{C}[W^{\mathbb{C}}]$ by multiplying $\phi^*(f)$, a degree 2 $K'_{\mathbb{C}}$-invariant polynomial.

By direct computation, $\overline{\phi(W^{\mathbb{C}})}$ has an open dense $K_{\mathbb{C}}$-orbit[16], say $\mathcal{O}$. On the other hand, $\operatorname{Ann}_{\mathcal{S}(\mathfrak{p}^+)} A$ contains the ideal defining the closed variety $\overline{\phi(W^{\mathbb{C}})}$, since $0 = \phi^*(f) \in \mathbb{C}[W^{\mathbb{C}}]$ if $f|_{\overline{\mathcal{O}}} = 0$. So we can view A as a $\mathbb{C}[\overline{\mathcal{O}}]$-module. Let $\mathscr{A}$ be the associated coherent sheaf of A on $\overline{\mathcal{O}}$. Fix an element $x \in \mathcal{O}$, we will calculate the fiber χ_x of $\mathscr{A}$ at x and show it is non-zero. Above discussion lead the first part of the main theorem of is section

Theorem 80. *(i) Let $L_{\widetilde{G}}(\mu)$ be the theta lifting of $\widetilde{G}'$-module $\tau^{\mu}_{\widetilde{G}'}$. Then*

$$\operatorname{AC}(L_{\widetilde{G}}(\mu)) = \dim \chi_x[\overline{\mathcal{O}}].$$

Here χ_x is the isotropy representation of $\varsigma^* \otimes L_{\widetilde{G}}(\mu)$ at $x \in \mathcal{O}$ under certain filtration.

[16] Note that G' does not have non-compact part, so we may define $\phi'\colon W^{\mathbb{C}} \to 0$. In this sense, $\mathcal{O}$ is the theta lift of the zero orbit.

(ii) If (G, G') is in the stable range with G' the smaller member,

$$\chi_x \cong \tau^* \circ \alpha = (\varsigma'^* \otimes \tau^{\mu}_{\widetilde{K}'}) \circ \alpha.$$

The homomorphism $\alpha\colon K_x \to K'_{\mathbb{C}}$ is defined by (4.17), (4.19) *and* (4.22) *for each cases.*

(iii) The K-spectrum equation (4.2),

$$A = \operatorname{Ind}_{K_x}^{K_{\mathbb{C}}} \chi_x,$$

holds except for $m \geq n$ in case $\mathbb{R}$; $m \geq n_1 = n_2$ in case $\mathbb{C}$; n is even and $2m \geq n$ in case $\mathbb{H}$.

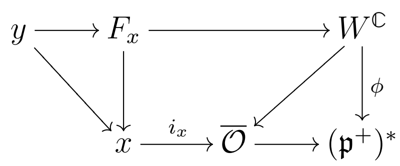
We first discusse when (4.2) holds. Let $\mathcal{N} = \phi^{-1}(\mathcal{O})$. Note that

$$\operatorname{Ind}_{K_x}^{K_{\mathbb{C}}} \chi_x = (\mathbb{C}[\mathcal{N}] \otimes \tau^*)^{K'_{\mathbb{C}}}.$$

Therefore, if $\partial\mathcal{N} = W^{\mathbb{C}} \backslash \mathcal{N}$ has codimension greater or equal to 2, $\mathbb{C}[W^{\mathbb{C}}] = \mathbb{C}[\mathcal{N}]$ via restriction map and (4.2) holds. On the other hand, $\partial\mathcal{N}$ has codimension 1 only happens when $\operatorname{Im}\phi$ contains invertible matrixes as described in (iii) of the theorem. In these cases, $f = 1/\phi^*(\det)$ is a $K'_{\mathbb{C}}$-invariant rational function in $\mathbb{C}[\mathcal{N}] \setminus \mathbb{C}[W^{\mathbb{C}}]$. So $\mathbb{C}[\mathcal{N}] \supset (\mathbb{C}[W^{\mathbb{C}}])[f]$. Hence $\operatorname{Ind}_{K_x}^{K_{\mathbb{C}}} \chi_x \supset (\mathbb{C}[W^{\mathbb{C}}] \otimes \tau^*)^{K'_{\mathbb{C}}}[f] \supsetneq A$.

Now we begin to calculate χ_x. The stable range case is given by (4.14). For non-stable range case, see (4.18), (4.20), (4.21), (4.23) and (4.24).

Consider following diagram



Let

$$\mathcal{I}_x = \left\{ \, x(f) - f \;\middle|\; f \in \mathfrak{p}^+ \, \right\} \mathcal{S}(\mathfrak{p}^+) \subset \mathcal{S}(\mathfrak{p}^+)$$

be the maximal ideal defining $x \in \mathcal{O} \subset (\mathfrak{p}^+)^*$. Let $F_x = \phi^{-1}(x)$ be the fiber of x in $W^{\mathbb{C}}$. Let $\mathbb{C}_x$ be the residual field at x isomorphic to $\mathbb{C}$. Since $\mathcal{O}$ is dense in $\phi(W^{\mathbb{C}})$, by Lemma 78

$$\mathbb{C}[F_x] = \mathbb{C}_x \otimes_{\mathcal{S}(\mathfrak{p}^+)} \mathbb{C}[W^{\mathbb{C}}] = \mathbb{C}[W^{\mathbb{C}}]/\phi^*(\mathcal{I}_x)\mathbb{C}[W^{\mathbb{C}}].$$

Since element in $\mathfrak{p}_x^+$ is $K'_{\mathbb{C}}$ invariant,

$$\begin{aligned}\chi_x =&\mathbb{C}_x \otimes_{\mathcal{S}(\mathfrak{p}^+)} A = A/\mathcal{I}_x A \\ &= \left(\mathbb{C}[W^{\mathbb{C}}] \otimes_{\mathbb{C}} \tau^*\right)^{K'_{\mathbb{C}}} / \mathcal{I}_x \left(\mathbb{C}[W^{\mathbb{C}}] \otimes_{\mathbb{C}} \tau^*\right)^{K'_{\mathbb{C}}} \\ &= \left(\mathbb{C}[W^{\mathbb{C}}] \otimes_{\mathbb{C}} \tau^* / \left(\phi^*(\mathcal{I}_x)\mathbb{C}[W^{\mathbb{C}}] \otimes_{\mathbb{C}} \tau^*\right)\right)^{K'_{\mathbb{C}}} \\ &\qquad \text{(taking } K'_{\mathbb{C}}\text{-invariant is an exact functor)} \\ &= \left((\mathbb{C}[W^{\mathbb{C}}]/\phi^*(\mathcal{I}_x)\mathbb{C}[W^{\mathbb{C}}]) \otimes_{\mathbb{C}} \tau^*\right)^{K'_{\mathbb{C}}} \\ &\qquad \text{(modules over } \mathbb{C} \text{ are flat)} \\ &= (\mathbb{C}[F_x] \otimes \tau^*)^{K'_{\mathbb{C}}}\end{aligned} \tag{4.13}$$

Remark:

1. In the stable range case and Tube type case, on can show that F_x has an single orbit under $K_x \times K'_{\mathbb{C}}$ action, i.e. F_x is a homogenous space. Fix a point $y \in F_x$ and let $S_y = \mathrm{Stab}_{K_{\mathbb{C}} \times K'_{\mathbb{C}}}(y)$ be the isotropy subgroup of y, then

$$\mathbb{C}[F_x] = \mathrm{Ind}_{S_y}^{K_x \times K'_{\mathbb{C}}} \mathbb{C}.$$

2. When (G, G') is in stable range, we define a map $\alpha\colon K_x \to K'_{\mathbb{C}}$ such that $S_y = K_x \otimes_\alpha K'_{\mathbb{C}}$. Then, by Lemma 79,

$$\chi_x = (\mathrm{Ind}_{S_y}^{K_x \times K'_{\mathbb{C}}} \mathbb{C} \otimes \tau^*)^{K'_{\mathbb{C}}} = \tau^* \circ \alpha \tag{4.14}$$

3. Outside stable range, when the symmetric domain is non-Tube type, F_x is not an single $K_x \times K'_{\mathbb{C}}$ orbit. Let L_x be the Levi subgroup of K_x. We will define an closed sub-variety $V_z \subset F_x$ such that $F_x = (L_x K'_{\mathbb{C}}) \cdot V_z$. We will show that V_z is a fiber of certain $L_x \times K'_{\mathbb{C}}$-equivariant fibration, and then calculate[17] the explicit form of χ_x under L_x-action.

4.4.2 Case by Case Computations

In this section, we calculate χ_x case by case. Fix an l-dimensional subspace $\mathbb{C}^l$ of $\mathbb{C}^n$, define $P_{l,n}$ to be the parabolic subgroup in $\mathrm{GL}(n, \mathbb{C})$ preserve $\mathbb{C}^l$, i.e.

$$\begin{aligned}P_{n,l} \cong& \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a \in \mathrm{GL}(l, \mathbb{C}), b \in M_{n,n-l}, c \in \mathrm{GL}(n-l, \mathbb{C}) \right\} \\ =&(\mathrm{GL}(l, \mathbb{C}) \times \mathrm{GL}(n-l, \mathbb{C})) \ltimes N_{n,l},\end{aligned} \tag{4.15}$$

[17] One should be careful when applying "FrobeniusReciprocity": for $H < G$, it is not obvious that a function on a H-invariant subset can always be extended correctly to a function on G. That is why we use fibration to avoid this problem.

where $N_{n,l}$ is the unipotent radical of $P_{n,l}$. Define

$$\alpha_{n,l}\colon P_{n,l} \to \mathrm{GL}(l, \mathbb{C}) \tag{4.16}$$

to be the projection $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a$.

4.4.2.1 Case $\mathbb{R}$

We have

$$\begin{aligned}
&(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(m)) \\
&W^{\mathbb{C}} \cong M_{n,m}, && (\mathfrak{p}^+)^* \cong \mathrm{Sym}_n, \\
&\forall A \in W^{\mathbb{C}} \quad (k, k') \in K_{\mathbb{C}} \times K'_{\mathbb{C}} \\
&\phi(A) = AA^T && K_{\mathbb{C}} \times K'_{\mathbb{C}} \cong \mathrm{GL}(n, \mathbb{C}) \times \mathrm{O}(m, \mathbb{C}), \\
&(k, k') \cdot A = kAk'^{-1}.
\end{aligned}$$

Stable range $(n \geq m)$**:** Let

$$y = I_{n,m}, \qquad x = \phi(y) = \begin{pmatrix} I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Then,

$$\begin{aligned}
K_x &= \{\, g \in P_{n,m} \mid \alpha_{n,m}(g) \in \mathrm{O}(m, \mathbb{C}) \,\} \\
&\cong (\mathrm{O}(m, \mathbb{C}) \times \mathrm{GL}(n-m, \mathbb{C})) \ltimes N_{n,m},
\end{aligned}$$

and

$$\alpha = \alpha_{n,m}. \tag{4.17}$$

Non-stable range Tube type $(n < m)$**:** Let

$$y = I_{n,m}, \qquad x = \phi(y) = I_n.$$

Then

$$\begin{aligned}
K_x &\cong \mathrm{O}(n) \\
S_y &= \{\, (o, oo_1) \in \mathrm{GL}(n, \mathbb{C}) \times \mathrm{O}(m, \mathbb{C}) \mid o \in \mathrm{O}(n), o_1 \in \mathrm{O}(m-n) \,\} \\
&\cong \triangle \mathrm{O}(n) \times \mathrm{O}(m-n)
\end{aligned}$$

Therefore,

$$\chi_x = \left(\operatorname{Ind}_{S_y}^{S_x \times K'_{\mathbb{C}}} \mathbb{C} \otimes \tau^*\right)^{K'_{\mathbb{C}}} \cong (\tau^*)^{\mathrm{O}(m-n)}. \tag{4.18}$$

4.4.2.2 Case $\mathbb{C}$

We have

$$\begin{aligned}
&(G, G') = (\mathrm{U}(n_1, n_2), \mathrm{U}(m)), \\
&W^{\mathbb{C}} = M_{n_1,m} \times M_{n_2,m} && (\mathfrak{p}^+)^* = M_{n_1,n_2}, \\
&K_{\mathbb{C}} \cong \mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}), && K'_{\mathbb{C}} \cong \mathrm{GL}(m, \mathbb{C}), \\
&\forall (A, B) \in W^{\mathbb{C}} \quad (k, k') \in K_{\mathbb{C}} \times K'_{\mathbb{C}}, \\
&\phi((A, B)) = AB^T, && (k, k') \cdot (A, B) = (k_1 A k'^{-1}, k_2 B k'^T).
\end{aligned}$$

Stable range $(n_1, n_2 \geq m)$**:** Let

$$y = (I_{n_1,m}, I_{n_2,m}), \qquad x = \phi(y) = \begin{pmatrix} I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Then

$$\begin{aligned}
K_x &= \left\{ (g_1, g_2) \in P_{n_1,m} \times P_{n_2,m} \;\middle|\; \alpha_{n_1,m}(g_1) = ((\alpha_{n_2,m}(g_2))^{-1})^T \right\} \\
&\cong (\triangle \mathrm{GL}(m) \times \mathrm{GL}(n_1 - m) \times \mathrm{GL}(n_2 - m)) \ltimes (N_{n_1,m} \times N_{n_2,m})
\end{aligned}$$

and $\alpha\colon K_x \to \mathrm{GL}(m)$ is defined by

$$K_x \ni (g_1, g_2) \mapsto \alpha_{n_1,m}(g_1). \tag{4.19}$$

Non-stable range Tube type $(m \geq n_1 = n_2 = n)$**:** Let

$$y = (I_{n,m}, I_{n,m}), \qquad x = I_n.$$

Then

$$\begin{aligned}
K_x &= \left\{ (g, {g^{-1}}^T) \in \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \right\} \cong \triangle \mathrm{GL}(n, \mathbb{C}), \\
S_y &= \left\{ (g, {g^{-1}}^T, g g_1) \in K_{\mathbb{C}} \times K'_{\mathbb{C}} \;\middle|\; g \in \mathrm{GL}(n), g_1 \in \mathrm{GL}(m-n) \right\} \\
&\cong \triangle \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(m-n, \mathbb{C}).
\end{aligned}$$

Therefore

$$\chi_x \cong (\tau^*)^{\mathrm{GL}(m-n, \mathbb{C})}. \tag{4.20}$$

Non-stable range non-Tube type $(m, n_1 > n_2)$**:** Without loss of generality, we may assume $n_1 > n_2$. Let $x = I_{n_1,n_2}$,

$$V_z = \left\{ \left(\begin{pmatrix} I_{n_2} & \mathbf{0} \\ \mathbf{0} & y_0 \end{pmatrix}, I_{n_2,m} \right) \,\middle|\, y_0 \in M_{n_1-n_2,m-n_2} \right\}$$

and $\mathrm{pr}\colon M_{n_1,m} \times M_{n_2,m} \to M_{n_1,n_2} \times M_{n_2,m}$ be the map $(A, B) \mapsto (AB^T, B)$. Let $z = (I_{n_1,n_2}, I_{n_2,m})$. Then $V_z = \mathrm{pr}|_{F_x}^{-1}(z) \cong M_{n_1-n_2,m-n_2}$. Moreover,

$$\begin{aligned} K_x &= \left\{ (g, g_1) \in P_{n_1,n_2} \times \mathrm{GL}(n_2, \mathbb{C}) \,\middle|\, \alpha_{n_1,n_2}(g) = (g_1^{-1})^T \right\} \\ L_x &= \triangle \mathrm{GL}(n_2, \mathbb{C}) \times \mathrm{GL}(n_1 - n_2, \mathbb{C}). \end{aligned}$$

Clearly, pr is $L_x \times K'_{\mathbb{C}}$-equivariant and $F_x = \mathrm{pr}^{-1}(\mathcal{O}_z)$, where $\mathcal{O}_z$ is the $L_x \times K'_{\mathbb{C}}$-orbit of z. Then we have following diagram.

$$\begin{array}{ccccccccc} M_{n_1-n_2,m-n_2} & \xrightarrow{\cong} & V_z & \xhookrightarrow{i_{V_z}} & F_x & \hookrightarrow & M_{n_1,m} \times M_{n_2,m} & = & W^{\mathbb{C}} \\ & & \downarrow & \lrcorner & \downarrow\!\!\downarrow \, \mathrm{pr}|_{F_x} & \lrcorner & \downarrow \, \mathrm{pr} & & \\ & & z & \xhookrightarrow{i_z} & \mathcal{O}_z & \hookrightarrow & M_{n_1,n_2} \times M_{n_2,m} & & \end{array}$$

Now $\mathrm{pr}_*(\mathscr{O}_{F_x})$ is an $(\mathcal{O}_z, L_x)$-module (c.f. Section 2.6.3), whose fiber at z equals to $i_z^*(\mathrm{pr}|_{F_x})_*(\mathscr{O}_{F_x}) = (\mathrm{pr}|_{F_x})_* i_{V_z}^*(\mathscr{O}_{F_x}) = \mathbb{C}[V_z]$. Therefore

$$\mathbb{C}[F_x] = \mathrm{pr}_*(\mathscr{O}_{F_x})(\mathcal{O}_z) = \mathrm{Ind}_{S_z}^{L_x \times K'_{\mathbb{C}}} \mathbb{C}[V_z],$$

where

$$\begin{aligned} S_z = & \left\{ \begin{array}{r} (gg_1, (g^{-1})^T, gg_2,) \\ \in L_x \times K'_{\mathbb{C}} \end{array} \,\middle|\, \begin{array}{l} g \in \mathrm{GL}(n_2, \mathbb{C}), g_1 \in \mathrm{GL}(n_1 - n_2, \mathbb{C}), \\ g_2 \in \mathrm{GL}(m - n_2, \mathbb{C}) \end{array} \right\} \\ & \cong \triangle \mathrm{GL}(n_2, \mathbb{C}) \times \mathrm{GL}(n_1 - n_2, \mathbb{C}) \times \mathrm{GL}(m - n_2, \mathbb{C}). \end{aligned}$$

Hence, as L_x-module,

$$\chi_x = (\tau^* \otimes \mathbb{C}[M_{n_1-n_2,m-n_2}])^{\triangle \mathrm{GL}(m-n_2,\mathbb{C})} \tag{4.21}$$

where $\mathrm{GL}(n_1 - n_2) \times \mathrm{GL}(m - n_2)$ act on $\mathbb{C}[M_{n_1-n_2,m-n_2}] \cong \mathbb{C}[V_z]$ linearly.

4.4.2.3 Case $\mathbb{H}$

We have

$$\begin{aligned}
&(G, G') = (\mathrm{O}^*(2n), \mathrm{Sp}(m)), && \\
&W^{\mathbb{C}} \cong M_{n,2m}, && \mathfrak{p}^+ \cong \mathrm{Alt}_n, \\
&K_{\mathbb{C}} \cong \mathrm{GL}(n, \mathbb{C}), && K'_{\mathbb{C}} \cong \mathrm{Sp}(2m, \mathbb{C}), \\
&\forall A \in W^{\mathbb{C}}, \quad (k, k') \in K_{\mathbb{C}} \times K'_{\mathbb{C}}, && \\
&\phi(A) = AJ_{2m}A^T, && (k, k') \cdot A = kAk'^{-1},
\end{aligned}$$

where $J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ is the matrix defining the symplectic form of $\mathbb{C}^{2m}$.

Stable range $(n \geq 2m)$**:** Let

$$y = I_{n,2m} \qquad x = \phi(y) = \begin{pmatrix} J & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Then,

$$\begin{aligned}
K_x =& \{ \, g \in P_{n,2m} \mid \alpha_{n,2m}(g) \in \mathrm{Sp}(2m, \mathbb{C}) \, \} \\
\cong& (\mathrm{Sp}(2m, \mathbb{C}) \times \mathrm{GL}(n - 2m)) \ltimes N_{n,2m}
\end{aligned}$$

and

$$\alpha = \alpha_{n,2m}. \tag{4.22}$$

Non-stable range Tube type $(2l = n \leq 2m)$**:** Let

$$y = \begin{pmatrix} I_l & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_l \end{pmatrix}, \qquad x = \phi(y) = J_{2l}.$$

Then

$$\begin{aligned}
K_x \cong& \mathrm{Sp}(2l, \mathbb{C}), \\
S_y =& \{ \, (g, g\, g_1) \in K_{\mathbb{C}} \times K'_{\mathbb{C}} \mid g \in \mathrm{Sp}(2l, \mathbb{C}), g_1 \in \mathrm{Sp}(2m - 2l, \mathbb{C}) \, \} \\
\cong& \triangle \mathrm{Sp}(2l, \mathbb{C}) \times \mathrm{Sp}(2m - 2l, \mathbb{C}).
\end{aligned}$$

Therefore,

$$\chi_x \cong (\tau^*)^{\mathrm{Sp}(2m-2l, \mathbb{C})}. \tag{4.23}$$

Non-stable range non-Tube type $(2l + 1 = n < 2m)$**:** Choose the symplectic form on C^{2m} such that first l-coordinate and last l-coordinate pairs. Let

$$
\begin{aligned}
x &= \begin{pmatrix} J_{2l} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}, \\
V_z &= \left\{ \begin{pmatrix} I_l & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_l \\ \mathbf{0} & y_0 & \mathbf{0} \end{pmatrix} \;\middle|\; y_0 \in M_{1,2m-2l}. \right\}
\end{aligned}
$$

and $\mathrm{pr}\colon M_{2l+1,2m} \to M_{2l,2m}$ be the projection deleting last row of a the matrix. Let

$$
z = \mathrm{pr}(V_z) = \begin{pmatrix} I_l & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_l \end{pmatrix} \in M_{2l,2m}.
$$

Then $V_z = \mathrm{pr}|_{F_x}^{-1}(z) \cong \mathbb{C}^{2m-2l}$. Moreover,

$$
\begin{aligned}
K_x &\cong \{\, g \in P_{2l+1,2l} \mid \alpha_{2l+1,2l}(g) \in \mathrm{Sp}(2l,\mathbb{C}) \,\} \\
&\cong (\mathrm{Sp}(2l,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})) \ltimes N_{2l+1,2l} \\
L_x &\cong \mathrm{Sp}(2l,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}).
\end{aligned}
$$

Clearly, pr is $L_x \times K'_{\mathbb{C}}$-equivariant and $F_x = \mathrm{pr}^{-1}(\mathcal{O}_z)$, where $\mathcal{O}_z$ is the $L_x \times K'_{\mathbb{C}}$-orbit of z. Then we have following diagram.

$$
\begin{array}{ccccccccc}
\mathbb{C}^{2m-2l} & \xrightarrow{\cong} & V_z & \xhookrightarrow{i_{V_z}} & F_x & \hookrightarrow & M_{2l+1,2m} & = & W^{\mathbb{C}} \\
 & & \downarrow & \lrcorner & \downarrow \mathrm{pr}|_{F_x} & \lrcorner & \downarrow \mathrm{pr} & & \\
 & & z & \xhookrightarrow{i_z} & \mathcal{O}_z & \hookrightarrow & M_{2l,2m} & &
\end{array}
$$

Now $\mathrm{pr}_*(\mathscr{O}_{F_x})$ is an $(\mathcal{O}_z, L_x)$-module (c.f. Section 2.6.3), whose fiber at z equals to $\mathbb{C}[V_z]$. Therefore

$$
\mathbb{C}[F_x] = \mathrm{Ind}_{S_z}^{L_x \times K'_{\mathbb{C}}} \mathbb{C}[V_z],
$$

where

$$
\begin{aligned}
S_z &= \left\{ (g, g_1 g_2) \in L_x \times K'_{\mathbb{C}} \;\middle|\; \begin{array}{l} \alpha_{2l+1,2l}(g) = J_{2l}^T (g_1^{-1})^T J_{2l} \in \mathrm{Sp}(2l,\mathbb{C}), \\ g_2 \in \mathrm{Sp}(2m-2l,\mathbb{C}) \end{array} \right\} \\
&\cong \triangle \mathrm{Sp}(2l,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{Sp}(2m-2l,\mathbb{C}).
\end{aligned}
$$

Finally, we have, as L_x-module

$$
\chi_x \cong (\tau^* \otimes \mathbb{C}[\mathbb{C}^{2m-2l}])^{\triangle \mathrm{Sp}(2m-2l,\mathbb{C})} \tag{4.24}
$$

where $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{Sp}(2m-2l)$ act on $\mathbb{C}[\mathbb{C}^{2m-2l}] \cong \mathbb{C}[V_z]$ linearly.

4.5 Isotropy representations of theta lifts of unitary characters

Let (G, G') be a non-compact real reductive dual pair in the stable range with G' the smaller member. We will study the associated cycle, isotropy representations and equation (4.2) for theta lifts of genuine unitary characters of G' in this section.

Let ρ' be a genuine unitary character of $\widetilde{G}'$ and

$$\nu \colon \mathscr{Y} \to \theta(\rho') \otimes \rho' \cong \theta(\rho')$$

be the unique (up to scaler) non-zero quotient map. $\mathrm{Gr}\,\theta(\rho')$ be the graded module of $\theta(\rho')$ under the natural filtration defined in Section 4.2. Now $\mathrm{Gr}\,\theta(\rho')$ is a $(\mathcal{S}(\mathfrak{p}), \widetilde{K})$-module. Note that the action of $\widetilde{K}$ and $\widetilde{K}'$ on the Fock space $\mathbb{C}[W^{\mathbb{C}}]$ is are just certain twist of the linear action by characters, say $\varsigma \otimes \varsigma'$, where ς (resp. ς') is a genuine character of $\widetilde{K}$ (resp. $\widetilde{K}'$). Define a $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module[18]

$$A := \mathrm{Gr}\,\theta(\rho') \otimes \varsigma^*. \tag{4.25}$$

We may identify the underlying spaces of A and which of $\mathrm{Gr}\,\theta(\rho')$, so the $K_{\mathbb{C}}$-action on A is induced from the linear action on $\mathbb{C}[W^{\mathbb{C}}]$ and $\mathcal{S}(\mathfrak{p})$-action on A is same as on $\mathrm{Gr}\,\theta(\rho')$.

Recall the double fibration (c.f. Section 2.3.7.3)

$$\mathfrak{p}^* \xleftarrow{\ \phi\ } W^{\mathbb{C}} \xrightarrow{\ \phi'\ } \mathfrak{p}'^*.$$

Since we are in the stable range, the null-cone $\phi'^{-1}(0)$ has an open $K_{\mathbb{C}}$-orbit $\mathcal{N}$ and

$$\mathcal{O} = \phi(\mathcal{N}) = \theta(\{\,0\,\})$$

is the theta lifts of the $\{\,0\,\}$-orbit of G'. We are going to prove following theorem.

Theorem 81. *Fix a point $x \in \mathcal{O}$. Let $K_x = \mathrm{Stab}_{K_{\mathbb{C}}}(x)$ be the isotropy subgroup of x. Then the isotropy representation of A at point x is isomorphic to*

$$\chi_x = (\varsigma' \otimes \rho'^*) \circ \alpha,$$

[18] $\widetilde{K}$ action factor through K. Then holomorphic extend the K-action to $K_{\mathbb{C}}$.

where $\alpha\colon K_x \to K'_{\mathbb{C}}$ *is certain surjective group homomorphism. In particular, the associated cycle*

$$\mathrm{AC}(\theta(\rho')) = [\overline{\mathcal{O}}].$$

Moreover, as $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$*-module,*

$$A \cong \mathrm{Ind}_{S_x}^{K_{\mathbb{C}}} \chi_x,$$

where the $\mathcal{S}(\mathfrak{p})$ *action on the right hand side is via the restriction map* $\mathcal{S}(\mathfrak{p}) = \mathbb{C}[\mathfrak{p}^*] \to \mathbb{C}[\mathcal{O}] = \mathbb{C}[K_{\mathbb{C}}/S_x]$.

Remark:

1. One can define a double covering $\varpi\colon \widetilde{K}_{\mathbb{C}} \to K_{\mathbb{C}}$ extending the covering $\widetilde{K} \to K$. Let $\widetilde{K}_x$ be the preimage of K_x. Then,

(i) $\widetilde{K}_x$ is the isotropy subgroup of x in $\widetilde{K}_{\mathbb{C}}$;

(ii) by (4.25), the isotropy representation of $\widetilde{K}_x$ isomorphic to

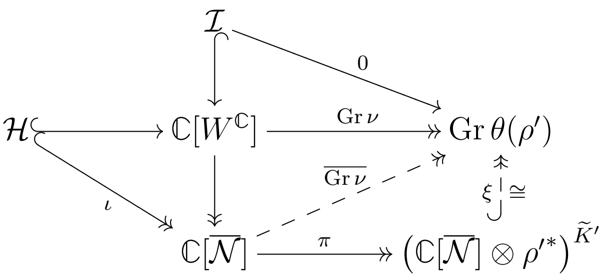
$$\widetilde{\chi}_x = \chi_x \circ \varpi \otimes \varsigma;$$

(iii) as $(\mathcal{S}(\mathfrak{p}), \widetilde{K}_{\mathbb{C}})$-module,

$$\mathrm{Gr}\,\theta(\rho) \cong \left(\mathrm{Ind}_{\widetilde{K}_x}^{\widetilde{K}_{\mathbb{C}}}(\chi_x \circ \varpi)\right) \otimes \varsigma = \mathrm{Ind}_{\widetilde{K}_x}^{\widetilde{K}_{\mathbb{C}}} \widetilde{\chi}_x.$$

2. In Section 4.6, we will explicitly give α, ς, ς' and so calculate χ_x for reductive dual pairs in Table 4.2, since results in Section 4.6 are build on the results of these cases.

In the rest of this section we are going to prove Theorem 81 by showing: as $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module, A isomorphic to the space of regular sections of an algebraic line bundle on $\mathcal{O}$.



(4.26)

Consider the diagram (4.26). Here

$$\mathcal{I} := \phi^*(\mathfrak{p}')\mathbb{C}[W^{\mathbb{C}}]$$

is the ideal generated by $\mathfrak{p}'$ in $\mathbb{C}[W^{\mathbb{C}}]$ and $\mathcal{H}$ is the space of harmonics of K'. Since (G, G') is in the stable range, $\mathcal{I}$ is precisely the ideal of polynomials vanishing on the null cone $\overline{\mathcal{N}} = \phi'^{-1}(0)$, i.e. the middle-vertical line in (4.26) is exact. On the other hand, $\mathfrak{p}' \subset \mathfrak{g}'$ acts trivially[19] on ρ'. Therefore it act trivially on $\mathrm{Gr}\,\theta(\rho')$ and $\mathrm{Gr}\,\nu$ factor through $\mathbb{C}[\overline{\mathcal{N}}]$. Denote $\overline{\mathrm{Gr}\,\nu}$ the corresponding map from $\mathbb{C}[\overline{\mathcal{N}}]$ to $\mathrm{Gr}\,\theta(\rho')$.

Lemma 82. *There is a $(\mathcal{S}(\mathfrak{p}), \widetilde{K})$-module isomorphism*

$$\xi\colon \left(\mathbb{C}[\overline{\mathcal{N}}] \otimes \rho'^*\right)^{\widetilde{K}'} \to \mathrm{Gr}\,\theta(\rho').$$

Moreover, let $\mathcal{I}_{\overline{\mathcal{O}}} \subset \mathcal{S}(\mathfrak{p})$ be the ideal of regular functions vanishing on $\overline{\mathcal{O}}$. Then $\mathcal{I}_{\overline{\mathcal{O}}} \subset \mathrm{Ann}_{\mathcal{S}(\mathfrak{p})}\mathrm{Gr}\,\theta(\rho')$.

Proof. Since $\widetilde{K}'$ act on $\theta(\rho') \otimes \rho'$ by ρ', $\overline{\mathrm{Gr}\,\nu}$ factor through the ρ' coinvariants of $\widetilde{K}'$. Therefore $\overline{\mathrm{Gr}\,\nu}$ factor through the $(\mathcal{S}(\mathfrak{p}), \widetilde{K})$-equivariant projection

$$\pi\colon \mathbb{C}[\overline{\mathcal{N}}] \to \left(\mathbb{C}[\overline{\mathcal{N}}] \otimes \rho'^*\right)^{\widetilde{K}'}.$$

Let $\xi\colon \left(\mathbb{C}[\overline{\mathcal{N}}] \otimes \rho'^*\right)^{\widetilde{K}'} \to A$ be the $(\mathcal{S}(\mathfrak{p}), \widetilde{K})$-map such that $\xi \circ \pi = \overline{\mathrm{Gr}\,\nu}$.

Now we prove ξ is injective by $\widetilde{K}$-type comparison. Let $\mathcal{H}$ be the space of Harmonic with respect to $K_{\mathbb{C}}$. Recall that $\iota\colon \mathcal{H} \hookrightarrow \mathbb{C}[W^{\mathbb{C}}] \to \mathbb{C}[\overline{\mathcal{N}}]$ is an $\widetilde{K} \times \widetilde{M}'^{(1,1)}$-isomorphism and, as $\widetilde{K} \times \widetilde{M}'^{(1,1)}$-module, $\mathcal{H} = \bigoplus_\mu \tau^\mu_{\widetilde{K}} \otimes \tau^\mu_{\widetilde{M}'^{(1,1)}}$ is multiplicity free. On the other hand[20], a $\widetilde{K}$-type $\tau^\mu_{\widetilde{K}}$ occur in $\theta(\rho')$ if and only if $\tau^\mu_{\widetilde{M}'^{(1,1)}}$ has a $\widetilde{K}'$-quotient to ρ', and the $\mathrm{Hom}_{\widetilde{K}'}(\tau^\mu_{\widetilde{M}'^{(1,1)}}, \rho')$ is at most dimension one. Therefore, by the multiplicity freeness of $\theta(\rho')$,

$$\theta(\rho') \cong (\mathcal{H} \otimes \rho'^*)^{\widetilde{K}'} \cong (\mathbb{C}[\overline{\mathcal{N}}] \otimes \rho'^*)^{\widetilde{K}'}$$

as $\widetilde{K}$-module.

For the second statement, note that $f \in \mathcal{S}(\mathfrak{p})$ act on $\mathbb{C}[W^{\mathbb{C}}]$, by multiplying $\phi^*(f) = f \circ \phi$. Hence $\phi^*(f) \in \mathcal{I}$ kills $\mathrm{Gr}\,\theta(\rho)$ if $f \in \mathcal{I}_{\overline{\mathcal{O}}}$. □

Now twiste every terms in (4.26) by ς^* and we insist $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ act on $\mathbb{C}[W^{\mathbb{C}}]$ linearly in the following calculation. Define, a character of $K'_{\mathbb{C}}$,

$$\chi' = \varsigma' \otimes \rho'^*. \tag{4.27}$$

[19] Note that ρ' is unitary character, the split trous has trivial action.
[20] see Section 2.3.6.

Then Lemma 82 implies

$$A \cong (\mathbb{C}[\overline{\mathcal{N}}] \otimes \varsigma' \otimes \rho'^{*})^{\widetilde{K}'} = (\mathbb{C}[\overline{\mathcal{N}}] \otimes \chi')^{K'_{\mathbb{C}}}, \tag{4.28}$$

and A is an $\mathbb{C}[\overline{\mathcal{O}}]$-module. Let $\mathscr{A}$ be the associated coherent sheaf of A on $\overline{\mathcal{O}}$. For a point $x \in \mathcal{O}$, let $\mathcal{I}_x \subset \mathcal{S}(\mathfrak{p})$ be the maximal ideal correspond to x and $F_x = \phi^{-1}(x) \cap \overline{\mathcal{N}}$ be its fiber in $\overline{\mathcal{N}}$. Fixing an element $y \in F_x$, let $S_y = \mathrm{Stab}_{K_{\mathbb{C}} \times K'_{\mathbb{C}}}(y)$ be the isotropy subgroup of y. One can easily verify case by case of following fact[21].

Lemma 83. *The fiber F_x is a single K_x-orbit. Moreover, there is a surjective homomorphism $\alpha\colon K_x \to K'_{\mathbb{C}}$ such that*

$$S_y = K_x \times_\alpha K'_{\mathbb{C}} = \{\, (k, \alpha(x)) \in K_x \times K'_{\mathbb{C}} \,\}.$$

Above definitions are summarized in diagram (4.29).

$$\begin{array}{ccccccccc} (S_x \times K'_{\mathbb{C}})/S_y & = & F_x & \hookrightarrow & \mathcal{N} & \hookrightarrow & \overline{\mathcal{N}} & \hookrightarrow & W^{\mathbb{C}} \\ & & \downarrow & & \lrcorner & & \downarrow{\scriptstyle \phi|_{\overline{\mathcal{N}}}} & & \downarrow{\scriptstyle \phi} \\ & & x & \hookrightarrow & \mathcal{O} & \overset{i_{\mathcal{O}}}{\hookrightarrow} & \overline{\mathcal{O}} & \hookrightarrow & \mathfrak{p}^* \end{array} \tag{4.29}$$

Since x has an open dense orbit $\mathcal{O}$ in $\overline{\mathcal{O}}$, the scheme theoretic fiber $\mathrm{Spec}\left(\mathbb{C}[\overline{\mathcal{N}}]/\phi^*(\mathcal{I}_x)\mathbb{C}[\overline{\mathcal{N}}]\right)$ is reduced and equal to $\mathbb{C}[F_x]$ by the Lemma 78. Hence the isotropy representation

$$\begin{aligned} \chi_x =& A/\mathcal{I}_x A = \left(\mathbb{C}[\overline{\mathcal{N}}] \otimes \chi'\right)^{K'_{\mathbb{C}}} / \left(\mathcal{I}_x \left(\mathbb{C}[\overline{\mathcal{N}}] \otimes \chi'\right)^{K'_{\mathbb{C}}}\right) \\ =& \left(\mathbb{C}[\overline{\mathcal{N}}]/\phi^*(\mathcal{I}_x)\mathbb{C}[\overline{\mathcal{N}}] \otimes \chi'\right)^{K'_{\mathbb{C}}} \\ =& (\mathbb{C}[F_x] \otimes \rho^*)^{K'_{\mathbb{C}}} \\ & \quad \text{(by the exactness of taking } K'_{\mathbb{C}}\text{-invariant, same argument as in (4.13))} \\ =& \left(\mathrm{Ind}_{K_x \times_\alpha K'_{\mathbb{C}}}^{K_x \times K'_{\mathbb{C}}} \mathbb{C} \otimes \chi'\right)^{K'_{\mathbb{C}}} \\ & \quad (F_x = (K_x \times K'_{\mathbb{C}})/(K_x \times_\alpha K'_{\mathbb{C}}) \text{ is a homogenous space}) \\ =& \chi' \circ \alpha. \\ & \quad \text{(by Lemma 79)} \end{aligned} \tag{4.30}$$

Proof of Theorem 81. We first show that, the associated cycle $\mathrm{AC}(\theta(\rho')) = [\overline{\mathcal{O}}]$. Form the second part of Lemma 82, $\mathrm{Supp}(A) \subset \overline{\mathcal{O}}$. Since the fiber of

[21] For the definition of α see also Section 2.3.7.3.

$\mathscr{A}$ at x, $\chi_x = \chi' \circ \alpha \neq 0$, $\mathcal{O} \subset \mathrm{Supp}(A)$. Hence $\mathrm{Supp}(A) = \overline{\mathcal{O}}$. Now the claim follows from the definition of associated cycle and $\dim \chi' = 1$.

Let $\mathscr{L}$ be the sheaf on $K_{\mathbb{C}}/K_x \cong \mathcal{O}$ associated with K_x-module χ_x. The natural map

$$\varrho\colon A \to \mathrm{Ind}_{K_x}^{K_{\mathbb{C}}} \chi_x = \mathscr{L}(\mathcal{O})$$

could be constructed explicitly by following evaluation map

$$h \mapsto (k \mapsto h(k \cdot y) \in \chi_x), \qquad \forall\, h \in A = (\mathbb{C}[\overline{\mathcal{N}}] \otimes \chi)^{K'_{\mathbb{C}}}, k \in K_{\mathbb{C}}.$$

It is an injection, since $K_{\mathbb{C}} K'_{\mathbb{C}} y = \mathcal{N} = \phi^{-1}(\mathcal{O})$ is an open dense subset in $\overline{\mathcal{N}}$. Now one can verify case by case that ϱ is an isomorphism by $K_{\mathbb{C}}$-spectrum comparison as in Yang's thesis [Yan11]. Furthermore,

$$\mathscr{A} = (i_{\mathcal{O}})_* \mathscr{L} \tag{4.31}$$

by Lemma 76. □

Remark:

1. There is a simpler way to check the $K_{\mathbb{C}}$-module isomorphism ϱ for a large class of dual pairs as following. Note that $\overline{\mathcal{N}}$ is a normal variety and the boundary of the open $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit $\mathcal{N}$ has codimension at least 2 in most of cases, which is true at least for pairs listed in Table 4.2 (for Case $\mathbb{R}$ we need $p, q > 2n$), c.f. Appendix of [NOZ06]. In these cases, $\mathbb{C}[\overline{\mathcal{N}}] = \mathbb{C}[\mathcal{N}] = \mathbb{C}[\phi^{-1}(\mathcal{O})]$. Therefore,

$$\begin{aligned} A = &\left(\mathbb{C}[\phi^{-1}(\mathcal{O})] \otimes \chi'\right)^{K'_{\mathbb{C}}} \\ =&\mathscr{A}(\mathcal{O}) = \mathscr{L}(\mathcal{O}) \\ =&\mathrm{Ind}_{K_x}^{K_{\mathbb{C}}} \chi_x. \end{aligned}$$

2. On the other hand, the validity of Theorem 81 implies that

$$A \cong (\mathbb{C}[\mathcal{N}] \otimes \chi')^{K'_{\mathbb{C}}} \tag{4.32}$$

always holds, although the boundary of $\mathcal{N}$ may not have codimension 2. We will use this observation in Section 4.6.

4.6 Isotropy representations of theta lifts of unitary lowest weight module

In this section, we study invariants of theta lifts of lowest weight modules. Unexplained notation is same as in Section 3.5.2.

We consider dual pairs listed in Table 4.2.

	$G^{p,q}$	G'	Stable range	Case I	$\operatorname{codim} \partial\mathcal{N}_0 \geq 2$
Case $\mathbb{R}$	$\mathrm{O}(p,q)$	$\mathrm{Sp}(2n,\mathbb{R})$	$p,q \geq 2n$ $\max\{p,q\} > 2n$	$r \geq n$	$n > r'$
Case $\mathbb{C}$	$\mathrm{U}(p,q)$	$\mathrm{U}(n_1,n_2)$	$p,q \geq n_1+n_2$	$r \geq n_1, n_2$	$\max\{n_1,n_2\} >$ $\min\{r',n_1,n_2\}$
Case $\mathbb{H}$	$\mathrm{Sp}(p,q)$	$\mathrm{O}^*(2n)$	$p,q \geq n$	$2r \geq n$	$n > 2r'$ or n is odd

Table 4.2: List of dual pairs II

4.6.1 Statment of the main theorem

Let $G = G^{p,q}$ such that the dual pair (G,G') is in stable range. Fix integers r, r' such that $r + r' = q$. Let $G_1 = G^{p,r}$ and $K^{r'} = G^{r',0}$. Let $K^{p,r} \cong G^{p,0} \times G^{0,r}$ be the maximal compact subgroup of G_1. Let $\mathfrak{g}_1 = \operatorname{Lie}(G_1)_{\mathbb{C}}$ and $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1$ be the non-compact part of $\mathfrak{g}_1$.

For a genuine $\widetilde{K}^{r'}$-representation $\tau^{\mu}_{\widetilde{K}^{r'}}$, let $L_{\widetilde{G}'}(\mu)$ be the theta lift of $\tau^{\mu}_{\widetilde{K}^{r'}}$ with respect to the compact dual pair $(K^{r'}, \widetilde{G}')$. Let $\theta^{p,q}(\rho')$ be the theta lifts of a genuine unitary $\widetilde{G}'$-character ρ' with respect to dual pair (G,G').

Recall the $(\mathfrak{g}_1, \widetilde{K}^{p,r})$-module in Definition 61

$$\theta^{p,r}(\mu) := \theta^{p,r}_{\rho'}(L_{\widetilde{G}'}(\mu)) \cong \left(\theta^{p,q}(\rho') \otimes \tau^{\mu}_{\widetilde{K}^{r'}}\right)^{\widetilde{K}^{r'}}. \tag{4.33}$$

It is the two-step theta lifts of $\tau^{\mu}_{\widetilde{K}^{r'}}$ up to the twisting of ρ'. The object of this section is to study the invariants of $\theta^{p,r}(\mu)$.

The natural filtration on $\theta^{p,q}(\rho')$ induced a filtration on $\theta^{p,r}(\mu)$. Lemma 74 shows it is a good filtration. As $(\mathcal{S}(\mathfrak{p}_1), \widetilde{K}^{p,r})$-module,

$$\operatorname{Gr} \theta^{p,r}(\mu) \cong (\operatorname{Gr} \theta^{p,q}(\rho') \otimes \tau^{\mu}_{\widetilde{K}^{r'}})^{\widetilde{K}^{r'}}.$$

Twisting with the genuine character $\varsigma^*|_{\widetilde{K}^{p,r}}$ as in Section 4.5, define a

$(\mathcal{S}(\mathfrak{p}_1), K_{\mathbb{C}}^{p,r})$-module

$$\begin{aligned} A(\mu) :=& \varsigma^*|_{\widetilde{K}^{p,r}} \otimes \theta^{p,r}(\mu) \cong (\varsigma^*|_{\widetilde{K}^{p,r}} \otimes \operatorname{Gr} \theta^{p,q}(\rho') \otimes \tau^{\mu}_{\widetilde{K}^{r'}})^{\widetilde{K}^{r'}} \\ \cong& (\varsigma^*|_{\widetilde{K}^{p,r}\widetilde{K}^{r'}} \otimes \operatorname{Gr} \theta^{p,q}(\rho') \otimes (\varsigma|_{\widetilde{K}^{r'}} \otimes \tau^{\mu}_{\widetilde{K}^{r'}}))^{\widetilde{K}^{r'}} \\ \cong& (A \otimes \tau)^{K_{\mathbb{C}}^{r'}}, \end{aligned} \tag{4.34}$$

where A is defined by (4.25) and

$$\tau = \varsigma|_{\widetilde{K}^{r'}} \otimes \tau^{\mu}_{\widetilde{K}^{r'}} \tag{4.35}$$

is a rational $K_{\mathbb{C}}^{r'}$-module.

Let

$$\operatorname{pr}_1 \colon \mathfrak{p}^* \to \mathfrak{p}_1^* \tag{4.36}$$

be the projection induced by $\mathfrak{p}_1 \hookrightarrow \mathfrak{p}$. Now we state the main theorem of this section.

Theorem 84. *Suppose that $\theta^{p,r}(\mu)$ is non-zero. Let $\overline{\mathcal{O}}$ be the associated variety of $\theta^{p,q}(\rho')$. Then*

(i) $\overline{\operatorname{pr}_1(\mathcal{O})}$ has an open dense $K_{\mathbb{C}}^{p,r}$-orbit, say $\mathcal{O}_1$, i.e. $\overline{\mathcal{O}}_1 = \overline{\operatorname{pr}_1(\mathcal{O})}$;

(ii) let $\mathcal{O}'$ be the dense $K'_{\mathbb{C}}$-orbit in the associated variety[22] of $L^*_{\widetilde{G}'}(\mu)$, then $\mathcal{O}_1$ is the theta lift of $\mathcal{O}'$ with respect to the dual pair $(G^{p,r}, G')$;

(iii) the associated variety

$$\mathcal{V}(\theta^{p,r}(\mu)) = \overline{\mathcal{O}}_1.$$

Suppose r satisfies "Case I" condition in Table 4.2. Fixing a point $x' \in \mathcal{O}'$, let $K'_{x'} = \operatorname{Stab}_{K'_{\mathbb{C}}}(x')$ be the isotropy subgroup of x, $\chi_{x'}$ be the isotropy representation of $L^*_{\widetilde{G}'}(\mu)$ (c.f. Theorem 80). Then

(iv) There is a point $x_1 \in \mathcal{O}_1$, a map $\beta \colon K_{x_1}^{p,r} \to K'_{x'} \hookrightarrow K'_{\mathbb{C}}$, such that[23]

$$\chi_{x_1} = (\chi_{x'} \otimes \chi') \circ \beta. \tag{4.37}$$

(v) In particular, the associated cycle

$$\operatorname{AC}(\theta^{p,r}_{\rho'}(L_{\widetilde{G}'}(\mu)) = \dim \chi_{x'}[\overline{\mathcal{O}_1}] = \theta(\operatorname{AC}(L^*_{\widetilde{G}'}(\mu))).$$

[22]For the associated variety of unitary lowest weight module see Section 4.4

[23]see (4.27) for the definition of χ'.

Under Case I, further assume "$\operatorname{codim}\partial\mathcal{N}_0 \geq 2$*" (see Table 4.2) is satisfied. Then*

(vi) as $(\mathcal{S}(\mathfrak{p}_1), K_{\mathbb{C}}^{p,r})$*-module*

$$A(\mu) \cong \operatorname{Ind}_{K_{x_1}^{p,r}}^{K_{\mathbb{C}}} \chi_{x_1}.$$

Remark:

1. Same as remark 1 after Theorem 81, twisting back $\varsigma|_{\widetilde{K}^{p,r}}$, one can get corresponding results of isotropy representation and K-spectrum equation of $\theta^{p,r}(\mu)$.

2. The condition $\operatorname{codim}\partial\mathcal{N}_0 \geq 2$ is exactly the condition for $L_{\widetilde{G}}(\mu)$ satisfies the K-spectrum equation.

3. Case I condition includes the cases that $(G^{p,r}, G')$ is in the stable range. But it also includes some non-stable range cases, for example, it is outside the stable range if $2n > r \geq n$ for Case $\mathbb{R}$.

4. When "Case I" condition is not satisfied we call it Case II. In Case II, $\chi_{x'}$ is more complicate. The calculation and results are similar to the non-Tube type domain case for unitary lowest weight modules (c.f. [Yam01] or Section 4.4).

5. We discuss the relationships between invariants of $L^*_{\widetilde{G}'}(\mu)$ and invariants of $L_{\widetilde{G}'}(\mu)$. We need a real Chevalley involution, see [Ada12]. There is an involution C on $\widetilde{G}'$ and $\mathfrak{g}'$ such which translate $L_{\widetilde{G}'}(\mu)$ into $L^*_{\widetilde{G}'}(\mu)$. Assume Siegel parabolic has form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. One could let $C(g) = \overline{g^T}$. It translate $\mathfrak{p}'^{+}$ into $\mathfrak{p}^-$. For any $x' \in \mathfrak{p}^+$, then $C(x') \in \mathfrak{p}^-$. Clearly $K'_{C(x')} = C(K'_{x'})$. Let $\lambda_{x'}$ be the isotropy representation of $L_{\widetilde{G}'}(\mu)$ at x'. The isotropy representations of $L^*_{\widetilde{G}'}(\mu)$ at $C(x')$ is given by $\lambda_{x'} \circ C$. In particular, $\operatorname{AC}(L^*_{\widetilde{G}'}(\mu)) = C(\operatorname{AC}(L_{\widetilde{G}'}(\mu)))$, where C translate $K'_{\mathbb{C}}$-orbits and preserve multiplicities.

4.6.2 proof of Theorem 84: general part

Now we begin to prove Theorem 84. We will first discus the part of proof holds for all pairs based on some geometry properties of moment maps. Then we supply data to justify these properties case by case in the next section.

Observe that, the pair (K, M') among diamond dual pairs of $(G^{p,q}, G')$ (c.f. [How89b], or Figure 2.1) is a direct sum of irreducible reductive dual pairs $(G^{p,0}, G') \oplus (G^{0,q}, G')$, we may write the $W^{\mathbb{C}} = W^+ \times W^-$ where

W^+ (resp. W^-) corresponds to pair $(G^{p,0}, G')$ (resp. $(G^{0,q}, G')$). Let $\phi'^+ : W^+ \to \mathfrak{p}^+$ (resp. $\phi'^- : W^- \to \mathfrak{p}^-$) be the corresponding moment map. The moment map[24]

$$\phi' : W^{\mathbb{C}} = W^+ \times W^- \to (\mathfrak{p}'^+)^* \times (\mathfrak{p}'^-)^* = \mathfrak{p}'^*$$

is given by $\phi'^+ \times \phi'^-$. Furthermore, $W^- = W_1^- \times W_2^-$, where W_1^- and W_2^- correspond to pair $(G^{0,r}, G')$ and $(G^{0,r'}, G')$ respectively. Let $\phi'^-_i : W_i^- \to (\mathfrak{p}^-)^*$ be corresponding moment maps, then $\phi'^- = \phi'^-_1 + \phi'^-_2$. The complex vector space associated with dual pair $(G^{p,r}, G')$ will be $W_1 = W^+ \times W_1^-$. Let $\phi_1 : W_1 \to \mathfrak{p}_1^*$ be the corresponding moment map.

We omit the case by case verification of part (i): $\mathrm{pr}_1(\overline{\mathcal{O}})$ has open $K^{p,r}$-orbit $\mathcal{O}_1$, since it will be clear when we explicitly write done an element $x_1 \in \mathcal{O}_1$ later.

Recall that $\mathcal{O}'$ is the open dense $K'_{\mathbb{C}}$-orbit in the associated variety of $L^*_{\widetilde{G}'}(\mu)$. By results in Section 4.4,

$$\overline{\mathcal{O}'} = \phi'^-_2(W_2^-).$$

Now we show that $\mathcal{O}_1$ is the theta lift of $\mathcal{O}'$. Consider commutative[25] diagram (4.38), where $\mathrm{pr} := \mathrm{id} \times \mathrm{pr}^-$ is the obvious projection and $\overline{\mathcal{N}}^+ = (\phi'^+)^{-1}(0)$ (resp. $\overline{\mathcal{N}}^- = (\phi'^-)^{-1}(0)$) is the null-cone in W^+ (resp. W^-).

$$\begin{array}{ccccc} W^+ \times W_1^- \times W_2^- = W^{\mathbb{C}} & \hookleftarrow & \overline{\mathcal{N}}^+ \times \overline{\mathcal{N}}^- & \xrightarrow{\phi} & \overline{\mathcal{O}} \\ \Big\downarrow{\scriptstyle \mathrm{pr}=\mathrm{id}\times\mathrm{pr}^-} & & \Big\downarrow{\scriptstyle \mathrm{pr}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_1} \\ W^+ \times W_1^- = W_1 & \hookleftarrow & \overline{\mathcal{N}}^+ \times \mathrm{pr}(\overline{\mathcal{N}}^-) & \xrightarrow{\phi_1} & \overline{\mathcal{O}}_1 \end{array} \qquad (4.38)$$

Note that

$$\overline{\mathcal{N}}^- = \left\{ (B_1, B_2) \in W_1^- \times W_2^- \;\middle|\; \phi'^-_1(B_1) + \phi'^-_2(B_2) = 0 \right\}. \qquad (4.39)$$

So

$$\mathrm{pr}^-(\overline{\mathcal{N}}^-) = (\phi'^-_1)^{-1}(-\phi'^-_2(W_2^-)) = (\phi'^-_1)^{-1}(\phi'^-_2(W_2^-)).$$

[24] G' is Hermitian symmetric, we have decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$

[25] the commutativity and surjectivity in the diagram could be verified by the explicit construction of moment map

Hence

$$\begin{aligned}\overline{\mathcal{O}}_1 = \mathrm{pr}_1(\overline{\mathcal{O}}) =& \mathrm{pr}_1 \circ \phi(\overline{\mathcal{N}}) = \phi_1 \circ (\mathrm{id} \times \mathrm{pr}^-)(\overline{\mathcal{N}}) \\ =& \phi_1(\overline{\mathcal{N}}^+ \times ({\phi'}_1^-)^{-1}(\overline{\mathcal{O}'})) \\ =& \phi_1(({\phi'}^+)^{-1}(0) \times ({\phi'}_1^-)^{-1}(\overline{\mathcal{O}'})) = \phi_1({\phi'_1}^{-1}(\overline{\mathcal{O}'})),\end{aligned}$$

i.e. $\mathcal{O}_1$ is the theta lift of $\mathcal{O}'$ by definition (c.f Definition 14).

Let

$$\mathcal{O}_0 := \mathrm{pr}_1^{-1}(\mathcal{O}_1) \cap \mathcal{O} \qquad \mathcal{N}_0 := \phi^{-1}(\mathcal{O}_0) \cap \mathcal{N} = (\mathrm{pr}_1 \circ \phi)^{-1}(\mathcal{O}_1) \cap \mathcal{N}.$$

Fix a element $x_1 \in \mathcal{O}_1$. We summarize above notations by commutative diagram[26] (4.40).

$$\begin{array}{ccccccccc} F_{x_1} & \longrightarrow & \mathcal{O}_0 & \longrightarrow & \mathcal{O} & \longrightarrow & \overline{\mathcal{O}} & \longrightarrow & \mathfrak{p}^* \\ \downarrow{\scriptstyle \mathrm{pr}_1} \lrcorner & & \downarrow{\scriptstyle \mathrm{pr}_1} \lrcorner & & \downarrow{\scriptstyle i_{\mathcal{O}}} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ x_1 & \longrightarrow & \mathcal{O}_1 & \longrightarrow & \overline{\mathcal{O}_1} & = & \overline{\mathcal{O}_1} & \longrightarrow & \mathfrak{p}_1^* \end{array} \quad (F_{x_1} \xrightarrow{i_{F_{x_1}}} \mathcal{O}) \tag{4.40}$$

Let $\mathcal{I}_{x_1}$ be the maximal ideal in $\mathcal{S}(\mathfrak{p}_1)$ defining x_1. Now the isotropy representation of $K_{x_1}^{p,r}$ is

$$\begin{aligned}\chi_{x_1} :=& A(\mu)/\mathcal{I}_{x_1}A(\mu) \\ & (\mathcal{I}_{x_1} \text{ is } K_{\mathbb{C}}^{r'} \text{ invariant and taking } K_{\mathbb{C}}^{r'}\text{-invariants is exact}) \\ =& (A/\mathcal{I}_{x_1}A \otimes \tau)^{K_{\mathbb{C}}^{r'}}\end{aligned} \tag{4.41}$$

with $A/\mathcal{I}_{x_1}A = ((i_{F_{x_1}}^*)\mathscr{L})(F_{x_1})$ by Lemma 77 and equation (4.31).

Before calculate χ_{x_1} explicitly, assume that:

(codim $\mathcal{N}_0 \geq 2$) the boundary $\partial\mathcal{N}_0 = \mathcal{N} \setminus \mathcal{N}_0$ has codimension at least 2 in $\mathcal{N}$.

We now show that the K-spectrum equation (4.2) holds under this condition. Note that $\mathcal{N}$ is smooth, by a long spectral sequence of local cohomology as in the proof of [CPS83, Theorem 4.4] or otherwise, $\mathbb{C}[\mathcal{N}] \cong \mathbb{C}[\mathcal{N}_0]$.

[26]The square with "⌟" in its center is a fiber product.

Now, by Theorem 39 of the equivalence of categories,

$$\begin{aligned} A =& (\mathbb{C}[\mathcal{N}] \otimes \chi')^{K'_{\mathbb{C}}} = (\mathbb{C}[\mathcal{N}_0] \otimes \chi')^{K'_{\mathbb{C}}} \\ =& \mathscr{A}(\mathcal{O}_0) = \mathscr{L}(\mathcal{O}_0) = \operatorname{Ind}_{K^{p,r}_{x_1}}^{K^{p,r}_{\mathbb{C}}}((i^*_{F_{x_1}})\mathscr{L})(F_{x_1}). \end{aligned}$$

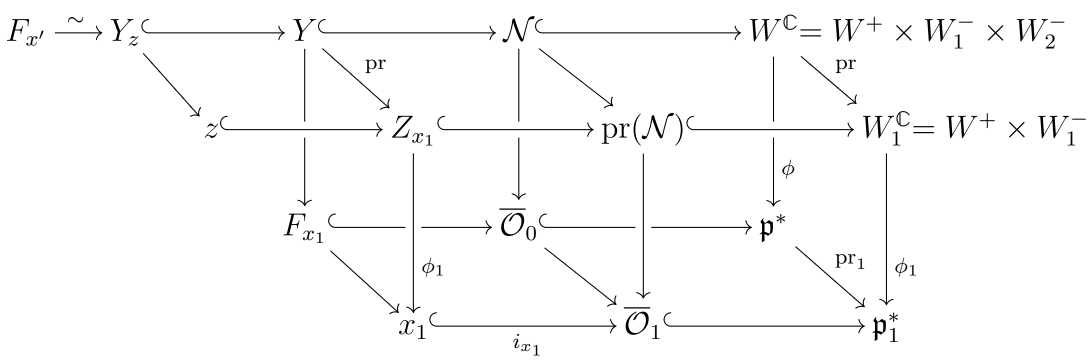
Therefore[27], as $(\mathcal{S}(\mathfrak{p}_1), K^{p,r}_{\mathbb{C}})$-module,

$$\begin{aligned} A(\mu) \cong& \left(\operatorname{Ind}_{K^{p,r}_{x_1}}^{K^{p,r}_{\mathbb{C}}}((i^*_{F_{x_1}})\mathscr{L})(F_{x_1}) \otimes \tau\right)^{K^{r'}_{\mathbb{C}}} \\ \cong& \operatorname{Ind}_{K^{p,r}_{x_1}}^{K^{p,r}_{\mathbb{C}}}\left(((i^*_{F_{x_1}})\mathscr{L})(F_{x_1}) \otimes \tau\right)^{K^{r'}_{\mathbb{C}}} \cong \operatorname{Ind}_{K^{p,r}_{x_1}}^{K^{p,r}_{\mathbb{C}}}\chi_{x_1}. \end{aligned}$$

An explicitly calculation[28] show that condition (codim ≥ 2) holds if r' satisfies the conditions listed in Table 4.2.

In the rest of this section, we describes the method calculating χ_{x_1}.

Case I: Consider following diagram.



Let Y be the fiber of x_1 for map $(\mathrm{pr}_1 \circ \phi)|_{\mathcal{N}}$ and Z_{x_1} be the fiber of x_1 for map $\phi_1|_{\mathrm{pr}(\mathcal{N})}$.

Since $A = (\mathbb{C}[\mathcal{N}] \otimes \chi')^{K'_{\mathbb{C}}}$,

$$A/\mathcal{I}_{x_1}A = (\mathbb{C}[Y] \otimes \chi')^{K'_{\mathbb{C}}}. \tag{4.42}$$

On the other hand, Z_{x_1} is a single $K^{p,r}_{x_1} \times K'_{\mathbb{C}}$-orbit, let $z \in Z_{x_1}$ with isotropy subgroup $S_z = \mathrm{Stab}_{K^{p,r}_{x_1} \times K'_{\mathbb{C}}} z$. Let $Y_z = \mathrm{pr}^{-1}(z) \cap Y$. So

$$\mathbb{C}[Y] \cong \operatorname{Ind}_{S_z}^{K^{p,r}_{x_1} \times K'_{\mathbb{C}}} \mathbb{C}[Y_z]. \tag{4.43}$$

Consider projection $\mathrm{pr}_2^- : W^- = W_1^- \times W_2^- \to W_2^-$. Note that in Case I, $\phi_1^- : W_1^- \to (\mathfrak{p}'^-)^*$ is a surjection. So $\mathrm{pr}_2^-(\overline{\mathcal{N}^-}) = \overline{\mathcal{O}'}$ by (4.39). Hence,

[27] By Theorem 37 (b)

[28] $\mathrm{codim}\partial\mathcal{N}_0$ equals to $n - r' + 1$ in Case $\mathbb{R}$; $n_2 + 1$ if $r' \leq n_2 \leq n_1$, $n_1 - n_2 + 1$ if $n_2 < n_1$ and $r' < n_1$ in Case $\mathbb{C}$; $2n - 4r' + 1$ if $2r' < n$, 3 if $n = 2l - 1$ $l < r, r'$ in Case $\mathbb{H}$.

$x' := \phi_2^-(z) \in \mathfrak{p}'^-$ generate $\mathcal{O}'$ under $K'_{\mathbb{C}}$-action. Let $F_{x'} := (\phi_2^-)^{-1}(x')$ be the fiber studied in Section 4.4. Then $Y_z = \{ z \} \times F_{x'}$. Moreover, there is a map

$$\beta \colon K_{x_1}^{p,r} \twoheadrightarrow K'_{x'} \hookrightarrow K'_{\mathbb{C}}$$

such that $S_z = K_{x_1}^{p,r} \times_\beta K'_{\mathbb{C}}$. Now, by (4.41), (4.42), (4.43) and Lemma 79,

$$\begin{aligned}
\chi_{x_1} &= \Big((\mathbb{C}[Y] \otimes \chi')^{K'_{\mathbb{C}}} \otimes \tau \Big)^{K_{\mathbb{C}}^{r'}} \\
&= \Big((\operatorname{Ind}_{S_z}^{K_{x_1}^{p,r} \times K'_{\mathbb{C}}} \mathbb{C}[Y_z] \otimes \tau)^{K_{\mathbb{C}}^{r'}} \otimes \chi' \Big)^{K'_{\mathbb{C}}} \\
&= \Big(\operatorname{Ind}_{S_z}^{K_{x_1}^{p,r} \times K'_{\mathbb{C}}} (\mathbb{C}[F_{x'}] \otimes \tau)^{K_{\mathbb{C}}^{r'}} \otimes \chi' \Big)^{K'_{\mathbb{C}}} \\
&= \Big(\operatorname{Ind}_{K_{x_1}^{p,r} \times_\beta K'_{\mathbb{C}}}^{K_{x_1}^{p,r} \times K'_{\mathbb{C}}} \chi_{x'} \otimes \chi' \Big)^{K'_{\mathbb{C}}} \\
&= (\chi_{x'} \otimes \chi') \circ \beta.
\end{aligned} \tag{4.44}$$

Here $\chi_{x'}$ is the isotropy representation of the twisted unitary lowest weight representation $\varsigma' \otimes L_{\widetilde{G}'}^{*}(\mu)$ (c.f. (4.13).).

Notice that the pair $(G^{0,r'}, G')$ correspond to the dual of the pair $(G^{r',0}, G')$ in Section 4.4. Our definition of τ in (4.35) is consistent with (4.12) if we "dual" everything simultaneously, especially the moment map ϕ_2^- maps W_2^- into $(\mathfrak{p}^-)^*$.

Case II: fix a point $x_1 \in \mathcal{O}_1$. Consider following commutative diagram

$$\begin{array}{ccccc}
Y & \longrightarrow & \mathcal{N} & \longrightarrow & W^{\mathbb{C}} \\
\big\downarrow & \lrcorner & \big\downarrow & & \big\downarrow \phi\circ\mathrm{pr} \\
x_1 & \longrightarrow & \overline{\mathcal{O}_1} & \longrightarrow & \mathfrak{p}_1^*
\end{array}$$

In this case, the fiber $\mathcal{N}_{x_1} := (\mathrm{pr}_1 \circ \phi|_{\mathcal{N}})^{-1}(x_1)$ will not be a single $K_{x_1}^{p,r} \times K_{\mathbb{C}}^{r'} \times K'_{\mathbb{C}}$-orbit. But there is a nice sub-variety $\mathcal{N}_s$ such that

$$\mathcal{N}_{x_1} = L_{x_1} K_{\mathbb{C}}^{r'} K'_{\mathbb{C}} \cdot \mathcal{N}_s,$$

where L_{x_1} is a Levi-subgroup of $K_{x_1}^{p,r}$. Moreover, $\mathcal{N}_s = \pi^{-1}(x_s)$ is the fiber[29] of some $L_{x_1} \times K_{\mathbb{C}}^{r'} \times K'_{\mathbb{C}}$-equivariant projection $\pi \colon \mathcal{N}_{x_1} \to M_s$ with $x_s \in M_s$ and $\mathcal{N}_s$ is isomorphic to the null-cone of a smaller dual pair[30] (G_s, G'_s). The isotropy subgroup $S_{x_s} = \mathrm{Stab}_{L_{x_1} \times K'_{\mathbb{C}} \times K_{\mathbb{C}}^{r'}} x_s$ isomorphic to $L_{x_1} \times K'_{s\mathbb{C}} \times K_{\mathbb{C}}^{r'-r}$,

[29] See commutative diagram (4.45)
[30] still in stable range

where $K_{\mathbb{C}}^{r'-r}$ and $K'_{s\mathbb{C}}$ are the compact part of the smaller dual pair.

$$\begin{array}{ccc} \mathcal{N}_s & \longrightarrow & \mathcal{N}_{x_1} \\ \big\downarrow & \lrcorner & \big\downarrow \pi \\ x_s & \longrightarrow & M_s \end{array} \tag{4.45}$$

Therefore,

$$A/\mathcal{I}_{x_1}A \cong (\mathbb{C}[\mathcal{N}_{x_1}] \otimes \chi)^{K'_{\mathbb{C}}} \cong (\mathrm{Ind}_{S_{x_s}}^{L_{x_1}\times K_{\mathbb{C}}^{r'}\times K'_{\mathbb{C}}} \mathbb{C}[\mathcal{N}_s] \otimes \chi')^{K'_{\mathbb{C}}}$$

and

$$\begin{aligned} \chi_{x_1} =& A(\mu)/\mathcal{I}_{x_1}A(\mu) = (A/\mathcal{I}_{x_1}A \otimes \tau)^{K_{\mathbb{C}}^{r'}} \\ =& \left((\mathrm{Ind}_{S_{x_s}}^{L_{x_1}\times K_{\mathbb{C}}^{r'}\times K'_{\mathbb{C}}} \mathbb{C}[\mathcal{N}_s] \otimes \chi')^{K'_{\mathbb{C}}} \otimes \tau \right)^{K_{\mathbb{C}}^{r'}} \\ =& \left((\mathbb{C}[\mathcal{N}_s] \otimes \chi'|_{K'_{s\mathbb{C}}})^{K'_{s\mathbb{C}}} \otimes \tau \right)^{K_{\mathbb{C}}^{r'-r}} \\ =& (A_s \otimes \tau)^{K_{\mathbb{C}}^{r'-r}}, \end{aligned} \tag{4.46}$$

where A_s is the twisted graded module of the theta lift of the same unitary character with respect to a smaller dual pair (G_s, G'_s) of same type.[31]

4.6.3 Proof of Theorem 84: case by case computation

4.6.3.1 Case $\mathbb{R}$: $(\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}))$

In this section, we let

$$\begin{aligned} &(G, G') = (\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R})), && G_1 = \mathrm{O}(p,r) \\ &K_{\mathbb{C}} = \mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(q,\mathbb{C}) && K'_{\mathbb{C}} = \mathrm{GL}(n,\mathbb{C}), \\ &K_{\mathbb{C}}^{p,r} = \mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(r,\mathbb{C}) && K_{\mathbb{C}}^{r'} = \mathrm{O}(r',\mathbb{C}) \\ &W^{\mathbb{C}} = M_{p,n} \times M_{q,n}, && W_1 = M_{p,n} \times M_{r,n}, \\ &\mathfrak{p}^* = M_{p,q}, \mathfrak{p}_1 = M_{p,r}, && \mathfrak{p}'^* = \mathrm{Sym}_n \times \mathrm{Sym}_n, \\ &\forall (A,B) \in M_{p,n} \times M_{q,n}, && (a,b,k') \in \mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(q,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}), \\ &\phi(A,B) = AB^T, && \phi'(A,B) = (A^TA, B^TB), \\ &\mathrm{pr}(A,B) = (A, B_1), && \mathrm{pr}_1(AB^T) = AB_1^T, \\ &\phi_1'^-(B) = B_1^TB_1, && \phi_2'^-(B) = B_2^TB_2 \end{aligned}$$

[31] "of same type" means still in the same family of dual pair listed in Table 4.2. Moreover the discriminate of G_s is same as G, hence we can identify the sets of genuine unitary characters of G' and G'_s naturally.

$(a, b, k') \cdot (A, B) = (aAk'^{-1}, bBk'^T),$

$\varsigma = \det^{\frac{n}{2}} \otimes \det^{-\frac{n}{2}}$ $\qquad \varsigma' = \det^{\frac{p-q}{2}}$

where $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ with $B_1 \in M_{r,n}$ and $B_2 \in M_{r',n}$. The $\widetilde{K} \times \widetilde{K}'$-action on the Fock-space $\mathscr{Y} \cong \mathbb{C}[W^{\mathbb{C}}]$ by

$$(((a, b), k') \cdot f)(A, B) = (\det^{\frac{n}{2}} a)(\det^{-\frac{n}{2}} b)(\det^{\frac{p-q}{2}} k') f(a^{-1}Ak', b^{-1}B(k'^{-1})^T) \quad \forall f \in \mathscr{Y}.$$

Now we calculate the isotropy representation of $\theta^{p,q}(\mathbf{1})$. Let

$$E_{p,n} = \begin{pmatrix} I_n \\ \mathbf{0}_{p-2n} \\ \sqrt{-1}I_n \end{pmatrix} \tag{4.47}$$

be the matrix formed by n linearly independent null column vectors. Let

$$y = (E_{p,n}, E_{q,n}) \in \mathcal{N}, \qquad x := \phi(y) = E_{p,n}E_{q,n}^T$$

The $K_{\mathbb{C}}$-orbit $\mathcal{O} := K_{\mathbb{C}} \cdot x \subset \mathfrak{p}^*$ is an open dense subset of $\phi(\overline{\mathcal{N}})$, consisted of rank n matrixes in $M_{p,q}$ such that the column and row vectors are all null.

Let $P_{p,n} \subset \mathrm{O}(p, \mathbb{C})$ be the stabilizer of the isotropic subspace spanned by the columns of $E_{p,n}$. Then [32]

$$P_{p,n} \cong (\mathrm{GL}(n, \mathbb{C}) \times \mathrm{O}(p - 2n, \mathbb{C})) \ltimes N_{p,n},$$

with $N_{p,n}$ its unipotent radical. Let

$$\alpha_{p,n} \colon P_{p,n} \to \mathrm{GL}(n, \mathbb{C})$$

by quotient out of $\mathrm{O}(p-2n, \mathbb{C}) \ltimes N_{p,n}$. Similarly, define $P_{q,n}$ and $\alpha_{q,n}$. Now

$$K_x = \left\{ (o_1, o_2) \in P_{p,n} \times P_{q,n} \;\middle|\; \alpha_{p,n}(o_1) = ((\alpha_{q,n}(o_2))^{-1})^T \in \mathrm{GL}(n, \mathbb{C}) \right\} \tag{4.48}$$

Define

$$\alpha \colon K_x \to K'_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C}) \quad \text{by} \quad (o_1, o_2) \mapsto \alpha_{p,n}(o_1).$$

[32] by fix an isotropic subspace dual to the column space of $E_{p,n}$

Therefore, the isotropy representation

$$\chi_x \cong \det^{\frac{p-q}{2}} \circ \alpha \colon K_x \to \mathrm{GL}(1,\mathbb{C})$$

and

$$A = \varsigma^* \otimes \mathrm{Gr}\,\theta^{p,q}(\mathbf{1}) \cong \mathrm{Ind}_{K_x}^{K_\mathbb{C}}(\det^{\frac{p-q}{2}} \circ \alpha).$$

Now consider the theta lifts of lowest weight module Let E_1 be the first r rows of $E_{q,n}$ and E_2 be the last r' rows of $E_{q,n}$. In all the cases, $x_1 = \mathrm{pr}_1(x) = E_{p,n}E_1^T$ generate a dense $K_\mathbb{C}^{p,r}$-orbit in $\overline{\mathrm{pr}_1(\mathcal{O})}$.

Now we calculate isotropy representation χ_{x_1} case by case.

Case I $(r \geq n)$ $\phi_1'^-(\overline{\mathcal{N}^-})$ is surjection to Symmetric matrix of rank less or equal to $j = \min\{\, r', n \,\}$. Let

$$z = (A_z, B_{1,z}), \quad \text{where}^{33} \quad A_z = E_{p,n}, B_{1,z} = \begin{pmatrix} I_j & \mathbf{0} \\ \mathbf{0} & E_{r-j,n-j} \end{pmatrix}.$$

Now $x' = -B_1^T B_1 = -\begin{pmatrix} I_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$.

$$Y_z = \left\{\, (A_y, B_{1,y}, B_2) \;\middle|\; B_2^T B_2 = x' \,\right\} \cong F_{x'} \subset M_{r',n}.$$

Define

$$\beta \colon K_{x_1}^{p,r} \to K'_\mathbb{C} \quad \text{by} \quad (o_1, o_2) \mapsto \alpha_{p,n}(o_1).$$

Case II $(r < n)$ We will change of basis in $\mathbb{C}^p$ such that the first r-coordinate are isotropic and dual to the last r-coordinate. Let $x_1 = I_{p,r}$. Then x_1 generate open dense $K_\mathbb{C}^{p,r}$-orbit in $\mathrm{pr}_1(\overline{\mathcal{O}})$.

$$K_{x_1}^{p,r} = \{\, (o_1, o_2) \in P_{p,r} \times \mathrm{O}(r,\mathbb{C}) \mid \alpha_{p,r}(o_1) = o_2 \in \mathrm{O}(r,\mathbb{C}) \,\}$$

Let $L_{p,r} \cong \mathrm{GL}(r,\mathbb{C}) \times \mathrm{O}(p',\mathbb{C})$ be the Levi subgroup of $P_{p,r}$. Then

$$L_{x_1} = L_{p,r} \times_{\alpha_{p,r}} \mathrm{O}(r,\mathbb{C}) \cong \triangle\mathrm{O}(r) \times \mathrm{O}(p-2r) \tag{4.49}$$

is a Levi subgroup of $K_{x_1}^{p,r}$.

Consider the projection $\pi \colon \mathcal{N}_{x_1} \hookrightarrow W^\mathbb{C} \to M_{r,n} \times M_{r,q-r} = M^s$ by

$$M_{r,n} \times M_{p-r,n} \times M_{r,n} \times M_{q-r} \ni (\begin{pmatrix} A_1 \\ * \end{pmatrix}, \begin{pmatrix} * \\ B_2 \end{pmatrix}) \mapsto (A_1, A_1 B_2^T).$$

[33] See (4.47) for the definition of $E_{*,*}$

Now π is an $L \times K'_{\mathbb{C}} \times K^{r'}_{\mathbb{C}}$-equivariant map. Let,

$$x_s = (I_{r,n}, iI_{r,q-r}).$$

Then $\pi(\mathcal{N}_{x_1})$ is an $L_{x_1} \times K^{r'} \times K'_{\mathbb{C}}$-orbit of x_s,

$$\pi^{-1}(x_s) = \left\{ \left(\begin{pmatrix} I_r & \mathbf{0} \\ \mathbf{0} & A_s \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} I_r & \mathbf{0} \\ iI_r & \mathbf{0} \\ \mathbf{0} & B_s \end{pmatrix}^T \right) \middle| (A_s, B_s) \in \mathcal{N}_s \right\} \cong \mathcal{N}_s$$

where $\mathcal{N}_s$ is the null cone for pair

$$(G_s, G'_s) = (\mathrm{O}(p-2r, q-2r), \mathrm{Sp}(2(n-r), \mathbb{R}))\,.$$

Finally, the isotropic subgroup of x_s is

$$\begin{aligned} S_{x_s} \cong & \triangle\mathrm{O}(r,\mathbb{C}) \times \mathrm{O}(p-2r,\mathbb{C}) \times \mathrm{GL}(n-r,\mathbb{C}) \times \mathrm{O}(2-2r,\mathbb{C}) \\ \cong & L \times \mathrm{GL}(n-r,\mathbb{C}) \times \mathrm{O}(q-2r,\mathbb{C}). \end{aligned}$$

4.6.3.2 Case $\mathbb{C}$: $(\mathrm{U}(p,q), \mathrm{U}(n_1,n_2))$

In this section, we let

$$\begin{aligned}
&(G, G') = (\mathrm{U}(p,q), \mathrm{U}(n_1,n_2)), && G_1 = \mathrm{U}(p,r) \\
&K_{\mathbb{C}} = \mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C}) && K'_{\mathbb{C}} = \mathrm{GL}(n_1,\mathbb{C}) \times \mathrm{GL}(n_2,\mathbb{C}), \\
&K^{p,r}_{\mathbb{C}} = \mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(r,\mathbb{C}) && K^{r'}_{\mathbb{C}} = \mathrm{O}(r',\mathbb{C}) \\
&W^{\mathbb{C}} = M_{p,n_1} \times M_{p,n_2} \times M_{q,n_1} \times M_{q,n_2}, && W_1 = M_{p,n_1} \times M_{p,n_2} \times M_{r,n_1} \times M_{r,n_2}, \\
&\mathfrak{p} = M_{p,q} \times M_{q,p}, \mathfrak{p}_1 = M_{p,r} \times M_{r,p}, && \mathfrak{p}' = M_{n_1,n_2} \times M_{n_2,n_1}, \\
&\forall (A,B,C,D) \in W^{\mathbb{C}}, && (a,b,k'_1,k'_2) \in K_{\mathbb{C}} \times K'_{\mathbb{C}}, \\
&\phi(A,B,C,D) = (AC^T, DB^T), && \phi'(A,B,C,D) = (A^TB, D^TC), \\
&\mathrm{pr}(A,B,C,D) = (A,B,C_1,D_1), && \mathrm{pr}_1(AC^T, DB^T) = (AC_1^T, D_1B^T), \\
&\phi_1'^{-}(C,D) = D_1^TC_1, && \phi_2'^{-}(C,D) = D_2^TC_2 \\
&(a,b,k'_1,k'_2)\cdot(A,B,C,D) = (aAk_1'^{-1}, a^{T^{-1}}Bk_1'^T, bCk_1'^T, b^{T^{-1}}Dk_1'^{-1}), \\
&\varsigma = \det^{\frac{n_1-n_2}{2}} \otimes \det^{-\frac{n_1-n_2}{2}} && \varsigma' = \det^{\frac{p-q}{2}} \otimes \det^{-\frac{p-q}{2}}
\end{aligned}$$

where $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, $D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ with $C_1 \in M_{r,n_1}$, $C_2 \in M_{r',n_1}$, $D_1 \in M_{r,n_2}$, $D_2 \in M_{r',n_2}$.

Assume $p+q$ is even, we calculate the isotropic representation of $\theta^{p,q}(\mathbf{1})$.

Let

$$
\begin{aligned}
y :=& (\begin{pmatrix} A & B \end{pmatrix}, \begin{pmatrix} C & D \end{pmatrix}) = (I_{p,n_1+n_2}, I_{q,n_1+n_2}),\\
x :=& \phi(y) = (\begin{pmatrix} I_{n_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0}_{n_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}).
\end{aligned}
$$

The $K_\mathbb{C}$-orbit $\mathcal{O} := K_\mathbb{C} \cdot x \subset \mathfrak{p}^*$ is an open dense subset of $\phi(\overline{\mathcal{N}})$, consist of pairs of matrices of rank n_1 and n_2 respectively and multiply them in two ways both give 0.

Let $P_{p,n} \subset \mathrm{GL}(p,\mathbb{C})$ be the stabilizer of the span of first n coordinates.

$$
P_{p,n} \cong (\mathrm{GL}(n,\mathbb{C}) \times \mathrm{O}(p-2n,\mathbb{C})) \ltimes N_{p,n},
$$

with $N_{p,n}$ its unipotent radical. Let

$$
\alpha_{p,n_1+n_2} \colon P_{p,n_1+n_2} \to \mathrm{GL}(n_1+n_2,\mathbb{C}).
$$

Then

$$
K_x = \left\{ (g_1,g_2) \in P_{p,n_1+n_2} \times P_{q,n_1+n_2} \;\middle|\; \alpha_{p,n_1+n_2}(g_1) = (\alpha_{q,n_1+n_2}(g_2)^T)^{-1} \in K'_\mathbb{C} \right\}
$$

Define

$$
\alpha \colon K_x \twoheadrightarrow K'_\mathbb{C} \quad \text{by} \quad (g_1,g_2) \mapsto \alpha_{p,n_1+n_2}(g_1).
$$

Therefore, the isotropic representation

$$
\chi_x \cong (\det{}^{\frac{p-q}{2}} \otimes \det{}^{-\frac{p-q}{2}}) \circ \alpha \colon K_x \to \mathrm{GL}(1,\mathbb{C})
$$

and

$$
A = \varsigma^* \otimes \mathrm{Gr}\,\theta^{p,q}(\mathbf{1}) \cong \mathrm{Ind}_{K_x}^{K_\mathbb{C}}((\det{}^{\frac{p-q}{2}} \otimes \det{}^{-\frac{p-q}{2}}) \circ \alpha).
$$

Now consider the theta lifts of lowest weight module. Without loss of generality, we assume $n_1 \geq n_2$.

Case I ($r \geq n_1, n_2$) In this case, the map $\phi_1^- \colon (C_1, D_1) \mapsto C_1^T D_1$, is surjective to the space of matrices M_{n_1,n_2}. Let

$$j = \min\{\, r', n_1, n_2 \,\},$$
$$z = (A_z, B_z, C_{1,z}, D_{1,z}), \quad \text{where}$$
$$\begin{pmatrix} A_z & B_z \end{pmatrix} = I_{p,n_1+n_2}, \quad C_{1,z} = \begin{pmatrix} \mathbf{0} \\ I_{r-n_2+j,n_1} \end{pmatrix}, \quad D_{1,z} = I_{r,n_2}.$$

So

$$x' = -\,\phi_1^-(z) = -D_{1,z}^T C_{1,z} = -\begin{pmatrix} \mathbf{0}_{n_2-j} & \mathbf{0} \\ \mathbf{0} & I_{j,n_1-n_2+j}, \end{pmatrix}$$
$$Y_z = \left\{ \left(A_z, B_z, \begin{pmatrix} C_{1,z} \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{1,z} \\ D_2 \end{pmatrix} \right) \,\middle|\, C_2^T D_2 = x' \right\} \cong F_{x'} \subset \mathfrak{p}'^-.$$

Define

$$\beta\colon K_{x_1}^{p,r} \to K'_{\mathbb{C}} \quad \text{by} \quad (g_1, g_2) \mapsto \alpha_{p,n_1+n_2}(g_1) \in \mathrm{GL}(n_1,\mathbb{C}) \times \mathrm{GL}(n_2,\mathbb{C}).$$

Case II $(n_1, n_2 \geq r)$ For simplicity, we only consider $n_1, n_2 \geq r$. In this case, let

$$x_1 = \left(I_{p,r}, \begin{pmatrix} \mathbf{0}_r & I_r & \mathbf{0} \end{pmatrix} \right) \in M_{p,r} \times M_{r,q}.$$

Then x_1 generate the open $K_{\mathbb{C}}^{p,r}$-orbit in $\mathrm{pr}_1(\overline{\mathcal{O}})$. Now

$$K_{x_1}^{p,r} = \{\, (g, g_1) \in P_{p,2r} \times \mathrm{GL}(r,\mathbb{C}) \mid \alpha_{p,2r}(g) = \triangle g_1 \in \mathrm{GL}(2r,\mathbb{C}) \,\}.$$

The Levi subgroup of $K_{x_1}^{p,r}$,

$$L_{x_1} \cong \triangle \mathrm{GL}(r,\mathbb{C}) \times \mathrm{GL}(p-2r,\mathbb{C}).$$

Define

$$\pi\colon \mathcal{N}_{x_1} \hookrightarrow W^{\mathbb{C}} \to M_s := M_{2r,n_1} \times M_{2r,n_2} \times M_{2r,q-r} \times M_{2r,q-r},$$

by

$$\left(\begin{pmatrix} A_1 & B_1 \\ * & * \end{pmatrix}, \begin{pmatrix} * & * \\ C_2 & D_2 \end{pmatrix} \right) \mapsto (A_1, B_1, A_1 C_2^T, B_2 D_2^T).$$

Let

$$x_s = (I_{r,n_1}, I_{r,n_2}, iI_{r,q-r}, iI_{r,q-r})$$

Then

$$\pi^{-1}(x_s) = \left\{ \begin{pmatrix} I_r & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_r & \mathbf{0} \\ \mathbf{0} & A_s & \mathbf{0} & B_s \end{pmatrix} \begin{pmatrix} I_r & \mathbf{0} & I_r & \mathbf{0} \\ iI_r & \mathbf{0} & iI_r & \mathbf{0} \\ \mathbf{0} & C_s & \mathbf{0} & D_s \end{pmatrix} \middle| (A_s, B_s, C_s, D_s) \in \mathcal{N}_s \right\}$$

where $\mathcal{N}_s$ is the null cone with respect to pair

$$(G_s, G'_s) = (\mathrm{U}(p-2r, q-2r), \mathrm{U}(n_1 - r, n_2 - r)).$$

The isotropic subgroup of x_s is

$$\begin{aligned} S_{x_s} \cong & \triangle \mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(q-2r, \mathbb{C}) \times \mathrm{GL}(n_1 - r, \mathbb{C}) \\ & \times \mathrm{GL}(n_2 - r, \mathbb{C}) \times \mathrm{GL}(q-2r, \mathbb{C}) \\ \cong & L_{x_1} \times \mathrm{GL}(n_1 - r, \mathbb{C}) \times \mathrm{GL}(n_2 - r, \mathbb{C}) \times \mathrm{GL}(q-2r, \mathbb{C}). \end{aligned}$$

4.6.3.3 Case $\mathbb{H}$: $(\mathrm{Sp}(p,q), \mathrm{O}^*(2n))$

In this section, we let

$$\begin{aligned}
&(G, G') = (\mathrm{Sp}(p,q), \mathrm{O}^*(2n)), && G_1 = \mathrm{Sp}(p,r) \\
&K_{\mathbb{C}} = \mathrm{Sp}(2p, \mathbb{C}) \times \mathrm{Sp}(2q, \mathbb{C}) && K'_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C}), \\
&K_{\mathbb{C}}^{p,r} = \mathrm{Sp}(2p, \mathbb{C}) \times \mathrm{Sp}(2r, \mathbb{C}) && K_{\mathbb{C}}^{r'} = \mathrm{Sp}(2r', \mathbb{C}) \\
&W^{\mathbb{C}} = M_{2p,n} \times M_{2q,n}, && W_1 = M_{2p,n} \times M_{2r,n}, \\
&\mathfrak{p} = M_{2p,2q}, \mathfrak{p}_1 = M_{2p,2r}, && \mathfrak{p}' = \mathrm{Alt}_{n,n} \times \mathrm{Alt}_{n,n}, \\
&\forall (A,B) \in M_{2p,n} \times M_{2q,n}, && (a,b,k') \in \mathrm{Sp}(2p,\mathbb{C}) \times \mathrm{Sp}(2q,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}), \\
&\phi(A,B) = AB^T, && \phi'(A,B) = (A^T J_{2p} A, B^T J_{2q} B), \\
&\mathrm{pr}(A,B) = (A, B_1), && \mathrm{pr}_1(AB^T) = AB_1^T, \\
&\phi_1'^{-}(B) = B_1^T J_{2r} B_1, && \phi_2'^{-}(B) = B_2^T J_{2r'} B_2 \\
&(a,b,k') \cdot (A,B) = (aAk'^{-1}, bBk'^T), && \\
&\varsigma = \det^{\frac{n}{2}} \otimes \det^{-\frac{n}{2}} && \varsigma' = \det^{p-q}
\end{aligned}$$

where $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ with $B_1 \in M_{2r,n}$ and $B_2 \in M_{2r',n}$. The $\widetilde{K} \times \widetilde{K}'$-action on the Fock-space $\mathscr{Y} \cong \mathbb{C}[W^{\mathbb{C}}]$ by

$$(((a,b),k') \cdot f)(A,B) = (\det^{\frac{n}{2}} a)(\det^{-\frac{n}{2}} b)(\det^{p-q} k') f(a^{-1}Ak', b^{-1}B(k'^{-1})^T) \quad \forall f \in \mathscr{Y}.$$

Now we calculate the isotropic representation of $\theta^{p,q}(\mathbf{1})$. Temporally, let $J_{2q} = \begin{pmatrix} \mathbf{0} & I_q \\ -I_q & \mathbf{0} \end{pmatrix}$ be the symplectic form on $\mathbb{C}^2 q$,

$$y = (I_{2p,n}, I_{2q,n}) \in \mathcal{N}, \qquad x = \phi(y) = I_{2p,n} I_{2q,n}^T = \begin{pmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

The $K_{\mathbb{C}}$-orbit $\mathcal{O} := K_{\mathbb{C}} \cdot x \subset \mathfrak{p}^*$ is an open dense subset of $\phi(\overline{\mathcal{N}})$, consisted of rank n matrixes in $M_{2p,2q}$ such that the column and row vectors are all null. Let $P_{2p,n} \subset \mathrm{Sp}(2p, \mathbb{C})$ be the stabilizer of the isotropic subspace spanned by the columns of $I_{2p,n}$. Then [34]

$$P_{2p,n} \cong (\mathrm{GL}(n, \mathbb{C}) \times \mathrm{Sp}(2p - 2n, \mathbb{C})) \ltimes N_{2p,n},$$

with $N_{2p,n}$ its unipotent radical. Let

$$\alpha_{2p,n} \colon P_{2p,n} \to \mathrm{GL}(n, \mathbb{C})$$

by quotient out of $\mathrm{O}(p - 2n, \mathbb{C}) \ltimes N_{2p,n}$. Similarly, define $P_{2q,n}$ and $\alpha_{2q,n}$. Now

$$K_x = \left\{ (g_1, g_2) \in P_{2p,n} \times P_{2p,n} \;\middle|\; \alpha_{2p,n}(g) = (\alpha_{2q,n}^T)^{-1} \right\}.$$

Define

$$\alpha \colon K_x \to K'_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C}) \quad \text{by} \quad (g_1, g_2) \mapsto \alpha_{2p,n}(g_1).$$

Therefore, the isotropic representation

$$\chi_x \cong \det{}^{(p-q)} \circ \alpha \colon K_x \to \mathrm{GL}(1, \mathbb{C})$$

and

$$A = \varsigma^* \otimes \mathrm{Gr}\, \theta^{p,q}(\mathbf{1}) \cong \mathrm{Ind}_{K_x}^{K_{\mathbb{C}}}(\det{}^{(p-q)} \circ \alpha).$$

Now consider the theta lifts of lowest weight module.

Case I $2r \geq n$ In this case, $\phi_1'^{-} \colon B_1 \mapsto B_1^T J_{2r} B_1 \in \mathrm{Alt}_n$ is surjective. Let $j = \min\{ r', \lfloor n/2 \rfloor \}$. Then $2r - 2j \geq 2(n - 2j) \geq 0$. Choose the form on $\mathbb{C}^{2q} = \mathbb{C}^{2j} \oplus \mathbb{C}^{2r-2j} \oplus \mathbb{C}^{2r'}$ be $J_{2j} \oplus J_{2r-2j} \oplus J_{2r'}$. Let

$$z = (A_z, B_{1,z}), \quad \text{where} \quad A_z = I_{2p,n}, B_{1,z} = \begin{pmatrix} I_{2j} & \mathbf{0} \\ \mathbf{0} & I_{2r-2j,n-2j} \end{pmatrix}.$$

[34] by fix an isotropic subspace dual to the column space of $I_{2p,n}$

Now

$$\begin{aligned} x' =&\phi_2^-(z) = -\phi_1^-(z) = -B_{1,z}^T J B_{1,z} = \begin{pmatrix} J_{2j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \\ Y_z =& \left\{ (A_z, B_{1,z}, B_2) \;\middle|\; B_2^T J_{2r'} B_2 = x' \right\} \cong F_{x'} \subset M_{2r',n}. \end{aligned}$$

Note that $x_1 = AB_{1,z}^T$. Let $S_{B_{1,z}} = \mathrm{Stab}_{K'_{\mathbb{C}} \times \mathrm{Sp}(2r)}(B_{1,z})$. Then

$$K_{x_1}^{p,r} = \left\{ (g, g_1) \in P_{2p,n} \times \mathrm{Sp}(2r, \mathbb{C}) \;\middle|\; (\alpha_{2p,n}(g), g_1) \in S_{B_{1,z}} \right\},$$

and define

$$\beta\colon K_{x_1}^{p,r} \to K'_{x'} = \mathrm{Stab}_{K'_{\mathbb{C}}}(x') \hookrightarrow K'_{\mathbb{C}} \quad \text{by} \quad (g, g_2) \mapsto \alpha_{2p,n}(g).$$

Case II $2r < n$ In this case, $2r' > n$. Let the symplectic form on $\mathbb{C}^{2q}$ be $J_{2r} \oplus J_{2r} \oplus J_{2q-4r}$. Fix symplectic form on $\mathbb{C}^{2p}$ such that first $2r$-coordinates and last $2r$-coordinates pair. Then

$$x_1 = \begin{pmatrix} I_{2r} \\ \mathbf{0} \end{pmatrix}$$

generate the open dense $K_{\mathbb{C}}^{p,r}$-orbit in $\mathrm{pr}_1(\overline{\mathcal{O}})$. Now

$$K_{x_1}^{p,r} = P_{2p,2r} \times_{\alpha_{2p,2r}} \mathrm{Sp}(2r, \mathbb{C}).$$

Let $L_{2p,2r} \cong \mathrm{GL}(2r, \mathbb{C}) \times \mathrm{Sp}(2p-4r, \mathbb{C})$ be the Levi subgroup of $P_{2p,2r}$. The Levi subgroup of $K_{x_1}^{p,r}$,

$$\begin{aligned} L_{x_1} =& \{ (g_1, g_2) \in L_{2p,2r} \times \mathrm{Sp}(2r, \mathbb{C}) \mid \alpha_{2p,2r}(g_1) = g_2 \} \\ \cong& \triangle \mathrm{Sp}(2r, \mathbb{C}) \times \mathrm{Sp}(2p - 4r, \mathbb{C}). \end{aligned}$$

Consider the projection $\pi\colon \mathcal{N}_{x_1} \hookrightarrow W^{\mathbb{C}} \to M_s := M_{2r,n} \times M_{2r,2q-2r}$ by

$$M_{2r,n} \times M_{2p-4r,n} \times M_{2r,n} \times M_{2q-2r,n} \ni \left(\begin{pmatrix} A_1 \\ * \end{pmatrix}, \begin{pmatrix} * \\ B_2 \end{pmatrix} \right) \mapsto (A_1, A_1 B_2^T).$$

Now π is an $L_{x_1} \times K'_{\mathbb{C}} \times K_{\mathbb{C}}^{r'}$-equivariant map. Let,

$$x_s = (I_{2r,n}, i I_{2r,2q-2r}).$$

Then $\pi(\mathcal{N}_{x_1})$ is an $L_{x_1} \times K^{r'} \times K'_{\mathbb{C}}$-orbit of x_s,

$$\pi^{-1}(x_s) = \left\{ \left(\begin{pmatrix} I_{2r} & \mathbf{0} \\ \mathbf{0} & A_s \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} I_{2r} & \mathbf{0} \\ iI_{2r} & \mathbf{0} \\ \mathbf{0} & B_s \end{pmatrix} \right) \middle| \, (A_s, B_s) \in \mathcal{N}_s \right\} \cong \mathcal{N}_s$$

where $\mathcal{N}_s$ is the null cone for pair

$$(G_s, G'_s) = (\mathrm{Sp}(2p - 4r, 2q - 4r), \mathrm{O}^*(2(n - 2r)))\,.$$

Finally, the isotropic subgroup of x_s is

$$\begin{aligned} S_{x_s} \cong & \triangle\mathrm{Sp}(2r, \mathbb{C}) \times \mathrm{Sp}(2p - 4r, \mathbb{C}) \times \mathrm{GL}(n - 2r, \mathbb{C}) \times \mathrm{Sp}(2q - 4r, \mathbb{C}) \\ \cong & L_{x_1} \times \mathrm{GL}(n - 2r, \mathbb{C}) \times \mathrm{Sp}(2q - 4r, \mathbb{C}). \end{aligned}$$

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